

Physical Computation of Nash Equilibrium: 'Hydraulic' Resource Selection

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*Dedicated to Dr. Bella Kessler, my high-school math teacher,
for continuously encouraging my nonstandard proofs. -Y.A.G.*

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Abstract

We present a novel construction, drawing intuition from a (physical) hydraulic system, constructively showing the existence of a strong Nash equilibrium in any resource selection game, the indifference of all players among Nash equilibria in such games, and the invariance of the load on each given resource across all Nash equilibria. The existence proof allows for explicit calculation of a Nash equilibrium and for explicit and direct calculation of the resulting (invariant) loads on resources, and does not hinge on any fixed-point theorem, on the Minimax Theorem or any equivalent result, or on the existence of a potential.

1 Notation

Definition 1 (Notation).

- (Naturals). We denote the natural numbers by $\mathbb{N} \triangleq \{0, 1, 2, \dots\}$.
- ($[n]$). For every $n \in \mathbb{N}$, we define $[n] \triangleq \{1, \dots, n\}$.
- (Reals). We denote the real numbers by \mathbb{R} .
- (Nonnegative Reals). We denote the nonnegative reals by $\mathbb{R}_{\geq} \triangleq \{r \in \mathbb{R} \mid r \geq 0\}$.
- (Maximizing Arguments). Given a set S and a function $f : S \rightarrow \mathbb{R}$ that attains a maximum value on S , we denote the *set* of arguments in S maximizing f by $\arg \text{Max}_{s \in S} f(s) \triangleq \{s \in S \mid f(s) = m\}$, where $m \triangleq \text{Max}_{s \in S} f(s)$.
- (Simplex). For a set $R \subseteq S$, we define

$$\Delta^R = \left\{ s \in [0, 1]^S \mid \sum_{j \in R} s_j = 1 \ \& \ \forall j \in S \setminus R : s_j = 0 \right\}.$$

(The set S will be clear from context.)

- (Nonempty Subsets). For a set S , we define $2_{\neq \emptyset}^S \triangleq 2^S \setminus \emptyset$ — the nonempty subsets of S .

Definition 2 (Plateau Height). Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a nondecreasing function. We say that $h \in \mathbb{R}$ is a *plateau height* of f if there exist $x \neq y \in \mathbb{R}$ s.t. $f(x) = f(y) = h$.

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2 Setting

Definition 3 (Resource Selection). Let $n \in \mathbb{N}$. An n -resource selection game is defined by a pair $((f_j)_{j=1}^n; (\mu^R)_{R \in 2_{\neq \emptyset}^{[n]}})$, where $f_j : \mathbb{R}_{\geq} \rightarrow \mathbb{R}$ is a nondecreasing function for every $j \in [n]$, and $\mu^R \in \mathbb{R}_{\geq}$ for every nonempty subset R of $[n]$.

In a resource selection game, each $R \in 2_{\neq \emptyset}^{[n]}$ indicates a player type, which may consume only from the resources in R ; μ^R is the amount to be consumed by all players of type R (one may imagine a continuum of such players, with total mass μ^R). For each resource $j \in [n]$, f_j is a function from the consumption amount of this resource to the consumption cost per resource unit. We now formally define these concepts.

Definition 4 (Consumption Profile/Nash Equilibrium). Let $G = ((f_j)_{j=1}^n; (\mu^R)_{R \in 2_{\neq \emptyset}^{[n]}})$ be a resource selection game.

1. A *consumption (strategy) profile* in G is a function $s : 2_{\neq \emptyset}^{[n]} \rightarrow \mathbb{R}_{\geq}^{[n]}$ s.t. $s(R) \in \mu^R \cdot \Delta^R$ for every $R \in 2_{\neq \emptyset}^{[n]}$.
2. Given a consumption profile s in G , we define $\mu_j^s \triangleq \sum_{R \in 2_{\neq \emptyset}^{[n]}} s_j(R)$ for every $j \in [n]$ — the load on (i.e. total consumption from) resource j . Furthermore, we define $h_j^s \triangleq f_j(\mu_j^s)$.
3. A *Nash equilibrium* in G is a consumption profile s s.t. for every $R \in 2_{\neq \emptyset}^{[n]}$, $h_k^s \leq h_j^s$ for every $k \in \text{supp}(s(R))$ and $j \in R$.

Example 1 (Home Internet / Cellular Market). Consider a scenario in which the resources are internet service providers (ISPs), and the players are customers on the market for home internet. (Alternatively, one could think of resources as cellular operators, and of players as customers on the market for cellular service.) Each customer may choose between the providers available in this customer's geographical area, and would like to get a connection with the largest bandwidth possible given this constraint. μ^R in this case is proportional to the amount of customers with possible ISPs R , and for each $j \in [n]$, we choose f_j s.t. $h_j^s = f_j(\mu_j^s)$ is inversely proportional to the effective bandwidth for each subscriber of ISP j , when there are μ_j^s subscribers to this ISP. If each ISP has the same total (i.e. overall) bandwidth, then the speed of the connection of a single customer subscribed to an ISP is inversely proportional to this ISP's number of subscribers, and so obtaining the fastest connection possible is equivalent to subscribing to a least-subscribed-to ISP, and so this case is captured by setting $f_j \triangleq \text{id}$ for every $j \in [n]$. Generalizing, we may imagine that e.g. some ISPs may have different total bandwidths than others (which may be captured by setting $f_j(x) \triangleq x/b_j$, where b_j is the total bandwidth of ISP j), or that some ISPs may even purchase some additional total bandwidth as their subscriber pool grows; in either scenario, in order to surf with greatest speed, each customer would prefer to subscribe not necessarily to a least-subscribed-to ISP (i.e. one with minimal μ_j^s), but rather to an ISP from whom the customer would receive the fastest connection, i.e. one with minimal $h_j^s = f_j(\mu_j^s)$.

The study of stability against group deviations was initiated by Aumann (1959), who considers deviations from which all deviators gain. Recently, the CS literature considers a considerably-stronger solution concept, according to which a deviation is considered beneficial even if only some of the participants in the deviating coalition gain, as long as none of the participants lose (see e.g. Rozenfeld and Tennenholtz (2006)). While stability against the classical all-gaining coalitional deviation is termed *strong equilibrium*, this more-demanding concept is referred to as *super-strong equilibrium*; there are very few results showing its existence in non-trivial settings. We now formally define both concepts.

Definition 5 (Strong / Super-Strong Nash Equilibrium). Let $G = ((f_j)_{j=1}^n; (\mu^R)_{R \in 2_{\neq \emptyset}^{[n]}})$ be a resource selection game and let s be a Nash equilibrium in G . For every $R \in 2_{\neq \emptyset}^{[n]}$ with $\mu^R > 0$, let $h^R \triangleq h_j^s$ for every $j \in \text{supp}(s(R))$. (h^R is well defined by definition of Nash equilibrium.)

- s is a *strong Nash equilibrium* if there exist no coalition $T \subseteq 2_{\neq \emptyset}^{[n]}$ and consumption profile $s' \neq s$ s.t. $s'|_{2_{\neq \emptyset}^{[n]} \setminus T} = s|_{2_{\neq \emptyset}^{[n]} \setminus T}$ and s.t. $h_k^{s'} < h^R$ for every $R \in T$ and $k \in \text{supp}(s'(R))$ s.t. $s'_k(R) > s_k(R)$.
- s is a *super-strong Nash equilibrium* if there exist no coalition $T \subseteq 2_{\neq \emptyset}^{[n]}$ and consumption profile s' s.t. $s'|_{2_{\neq \emptyset}^{[n]} \setminus T} = s|_{2_{\neq \emptyset}^{[n]} \setminus T}$ and s.t. $h_k^{s'} \leq h^R$ for every $R \in T$ and $k \in \text{supp}(s'(R))$, with a strict inequality for at least one such pair (R, k) .

Remark 1. Every super-strong Nash equilibrium is a strong Nash equilibrium.

3 Resource Selection via One-Way Communicating Vessels — A Special Case

Gonczarowski and Tennenholtz (2014) give a constructive proof for the very special case in which $\mu^R > 0$ only for player types R of the form $[k]$ for some $k \in [n]$ (i.e. $\mu^R = 0$ for every R not of this form); their proof draws its intuition from an analogy to a system of communicating vessels — see Fig. 1. We note that Kaminsky (2000), in turn, uses an analogy to quite a different system of communicating vessels to solve rationing problems; his motivation is quite different, and involves extending bilateral rationing rules. While Kaminsky uses a set of two-way communicating vessels, Gonczarowski and Tennenholtz use a set of one-way communicating vessels. In this context, the problem of finding a Nash equilibrium among players may be regarded as a rationing problem with constraints. The treatment of Gonczarowski and Tennenholtz (2014) also sheds new light on rationing problems, as resource selection games of sorts among a continuum of good-fragments.

4 Formal Results

Our goal is to *constructively* prove the following three theorems and corollary.

Theorem 1 (\exists Nash Equilibrium). *Let $G = ((f_j)_{j=1}^n; (\mu^R)_{R \in 2_{\neq \emptyset}^{[n]}})$ be a resource selection game. If f_1, \dots, f_n are continuous, then a Nash equilibrium exists in G .*

Theorem 2 (Resource Costs are Independent of Nash Equilibrium). *Let G be an n -resource selection game. $h_j^s = h_j^{s'}$ for every $j \in [n]$ and every two Nash equilibria s, s' in G .*

Corollary 1. *Let $G = ((f_j)_{j=1}^n; (\mu^R)_{R \in 2_{\neq \emptyset}^{[n]}})$ be a resource selection game.*

1. (Players are Indifferent between Nash Equilibria). *$h_k^s = h_k^{s'}$ for every $k \in \text{supp}(s(R))$ and $k' \in \text{supp}(s'(R))$, for every $R \in 2_{\neq \emptyset}^{[n]}$ and every two Nash equilibria s, s' in G .*
2. (Resource Loads are Independent of Nash Equilibrium). *If no two of $(f_j)_{j=1}^n$ share any plateau height, then $\mu_j^s = \mu_j^{s'}$ for every $j \in [n]$ and every two Nash equilibria s, s' in G .*

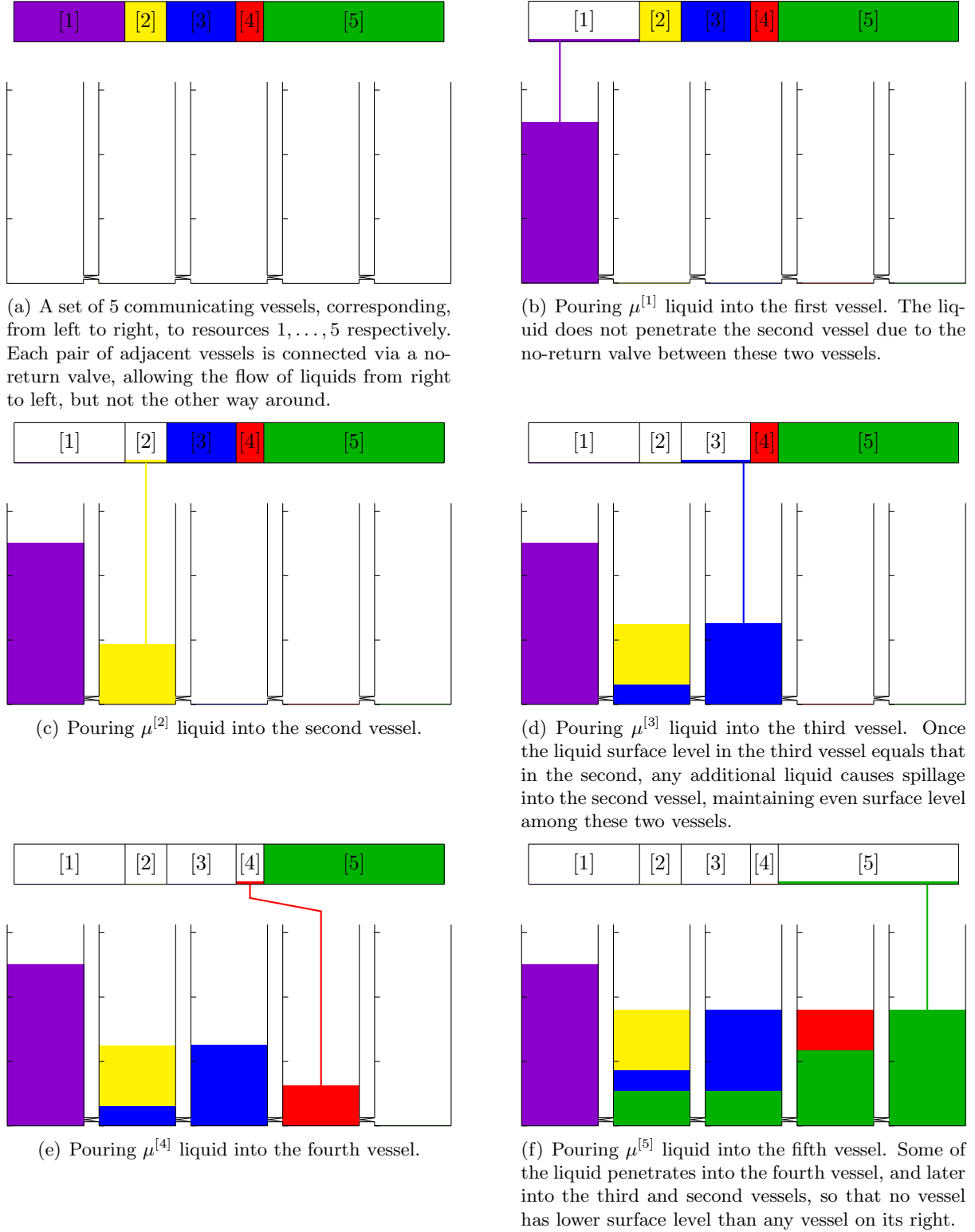


Figure 1: Illustration of the special-case construction of Gonczarowski and Tennenholtz (2014), for $n = 5$ and for $f_j = \text{id}$ for every $j \in [5]$. E.g. as exactly 80% of the blue liquid in Fig. 1(f) is in the third vessel and the remaining 20% is in the second one, the strategy for player type [3] in the Nash equilibrium that they construct is $0.8 \cdot \mu^{[3]}$ consumption from resource 3 and $0.2 \cdot \mu^{[3]}$ consumption from resource 2.

Theorem 3 (All Nash Equilibria are Strong / Super Strong). *Let $G = ((f_j)_{j=1}^n; (\mu^R)_{R \in 2^{[n]} \setminus \emptyset})$ be a resource selection game.*

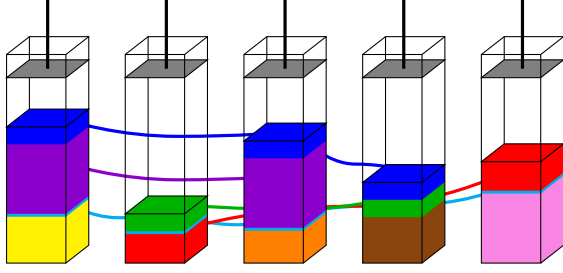
1. *All Nash equilibria in G are strong.*
2. *If h_j^s is not a plateau height of f_j for each $j \in [n]$ in any/every Nash equilibrium s , then all Nash equilibria in G are super strong.*

We emphasize that our proof of Theorem 1 allows for explicit calculation of a Nash equilibrium, and does not hinge on any fixed-point theorem, on the Minimax Theorem or any equivalent result, or on the existence of a potential. Furthermore, our proof of Theorem 2 gives an explicit formula for h_j^s for every $j \in [n]$.

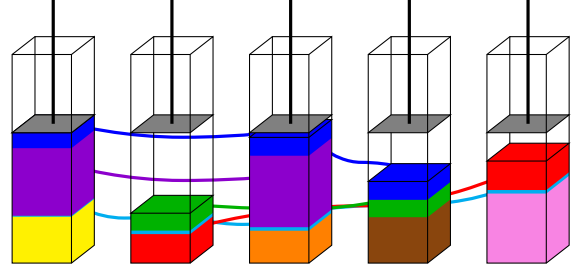
5 Construction and Hydraulic Intuition

In this section, we intuitively survey the construction underlying our results, as a prelude to the formal analysis given in Section 6. We start with the special case in which $f_j = \text{id}$ for every $j \in [n]$, i.e. $h_j^s = \mu_j^s$ for every $j \in [n]$ and consumption profile s . Our “hydraulic” construction for this case, from which our analysis draws intuition, is illustrated in Fig. 2. The intuition underlying our results draws from a number of key observations regarding this construction (we generalize and formalize these observations in Section 6):

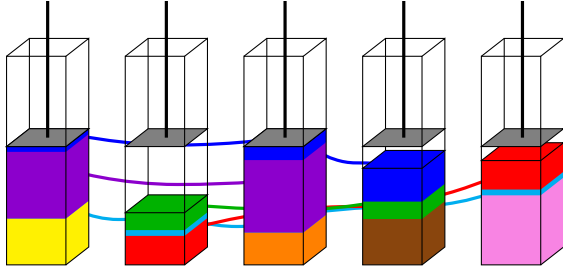
1. If the pistons in a set S (e.g. $S = \{1, 3\}$ or $S = \{4\}$) of containers stop simultaneously, then at the time of their stopping, no liquid under these pistons can escape to any container in which the piston has not yet stopped (or else it would do so and the piston above it would not stop).
2. By Observation 1, and as pistons that stop later in time stop at a lower height, in the resulting consumption profile no player type has any incentive to deviate, and so it is indeed a Nash equilibrium.
3. If we initially distribute the liquid of each “colour” (among the various balloons corresponding to this colour) according to some Nash equilibrium (e.g. if we initially distribute the liquid as in Fig. 2(f)), then the liquid distribution would not change during the entire process of descent of the pistons. Therefore, each Nash equilibrium may be attained from some initial liquid distribution.
4. After the first and third pistons (in Fig. 2) stop, we effectively start over, solving a 3-resource (2, 4, 5) selection game among all player types whose original acceptable resources were not merely the first and/or third resource.
5. Per Observation 1 above, no part of the liquid under the first and third pistons (in Fig. 2) when they stop can ever end up in any container other than these two, regardless of the initial liquid distribution. Therefore, the first and third pistons always stop having under them at least the liquid that is under them in Fig. 2(d), and therefore at this or a higher height. Per the same observation, the pistons stopping first always stop having under them solely liquid that cannot escape to other containers, and so if this set were not the first and third pistons, then it would stop below the stopping height of the first and third pistons, regardless of the initial liquid distribution. Therefore, the first and third pistons always stop first, and at the same height. Using Observation 4, an inductive argument can show that the height at which each piston stops (and the stopping order) is independent of the initial liquid distribution, and so by Observation 3, h_j^s for every $j \in [n]$ is independent



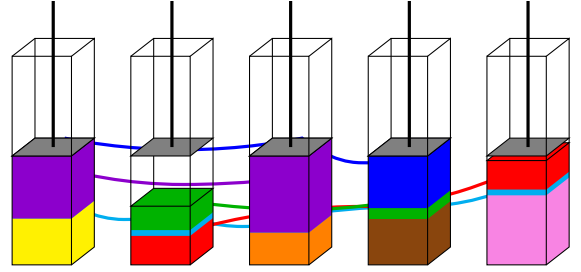
(a) A set of 5 open-top hollow box containers, corresponding, from left to right, to resources 1, ..., 5 respectively. For each player type R with $\mu^R > 0$ (each such type is assigned a distinct colour in the illustration), a balloon, or plastic bag, is placed in each container $j \in R$. Balloons corresponding to the same type R are connected via a thin tube emerging from a narrow slit (not shown) running vertically along the back of each container, and are jointly filled with μ^R liquid.



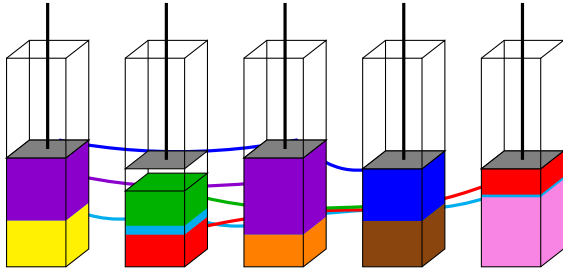
(b) Pistons are simultaneously lowered through the top sides of all container. As the piston in the first container reaches the balloons in this container, they are compressed, causing the balloons connected to them (i.e. the purple balloon in the third container, the blue balloons in the third and fourth containers, and the cyan balloons in the second, third and fifth containers) to inflate.



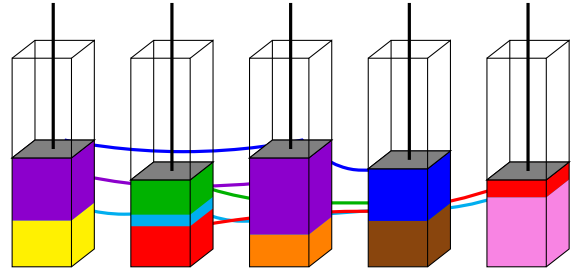
(c) As the piston in the third container reaches the balloons in this container, they start to compress as well, causing e.g. the interconnected blue balloon in the fourth container to inflate even faster.



(d) At a certain point in time, no balloon in the first or third containers can be compressed any further, as all the liquid in these containers that could have escaped to other containers has been depleted. The pistons in the first and third containers halt, and the remaining pistons continue their descent.



(e) At some later point in time, no balloon in the fourth container can be compressed any further, as all the liquid in this container that could have escaped to any container other than the first or the third ones has been depleted.



(f) Eventually, no balloon in the second or fifth container can be compressed any further, and the process concludes.

Figure 2: Illustration of the construction underlying our analysis, for $n = 5$ and for $f_j = \text{id}$ for every $j \in [5]$. E.g. as exactly 87.5% of the red liquid in Fig. 2(f) is in the second container and the remaining 12.5% is in the fifth one, the strategy for the player type corresponding to the red colour (i.e. $R = \{2, 5\}$) in the (super-)strong Nash equilibrium that we construct is $0.875 \cdot \mu^{\{2,5\}}$ consumption from resource 2 and $0.125 \cdot \mu^{\{2,5\}}$ consumption from resource 5; similarly, as all of the blue liquid is in the fourth container, the strategy for the “blue” type ($\{1, 3, 4\}$) in this equilibrium is $\mu^{\{1,3,4\}}$ consumption, solely from resource 2.

of the choice of Nash equilibrium. Furthermore, by the same argument, a player who consumes from more than one resource always consumes from resources with the same h_j^s , independently of the choice of Nash equilibrium s .

We note that while the final piston heights (i.e. values of h_j^s) are independent of the initial distribution of liquid among connected balloons (i.e. of the choice of Nash equilibrium s), the final liquid distribution (i.e. player strategies) is not; e.g. in Fig. 2(f), any amount of cyan liquid may be transferred from the second to the fifth container “in exchange for” an identical amount of red liquid.

For the general case of arbitrary f_j , we intuitively think of replacing the j ’th box container, for every $j \in [n]$, with a container shaped so that whenever it is filled with any amount $\mu_j \in \mathbb{R}_{\geq}$ of liquid, the resulting surface level would be precisely $f_j(\mu_j)$. See Fig. 3 for an illustration.

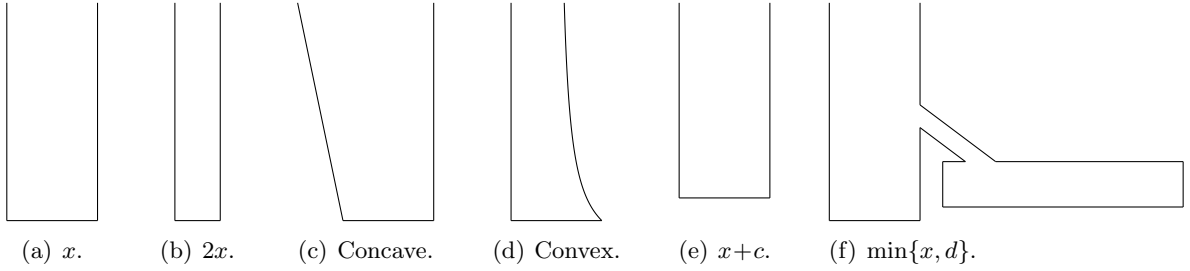


Figure 3: Containers corresponding to various functions f_j . (Assume that the container to the right of the vessel depicted in Fig. 3(f) is large enough so as to never fill up, yet may only be occupied by balloons as long as the piston does not pass the tube connecting this container to the main vessel.)

We emphasize that while the actual construction of such vessels requires differentiability of the functions f_j , our formal proof of Theorem 1 only requires continuity of these functions, while our formal proofs of Theorems 2 and 3 and Corollary 1 do not require even that.

6 Formal Derivation

We are now ready to formally present our analysis. Full proofs of all the results of this section and of Section 4 are given in Appendix A.

6.1 Communicating-Vessel Equalization

Let S be the set of first pistons to stop during the process depicted in Fig. 2. Assume that when these pistons stop, the total amount of liquid in the respective containers is μ . At what height did the pistons stop? In this section we formalize the answer to this question.

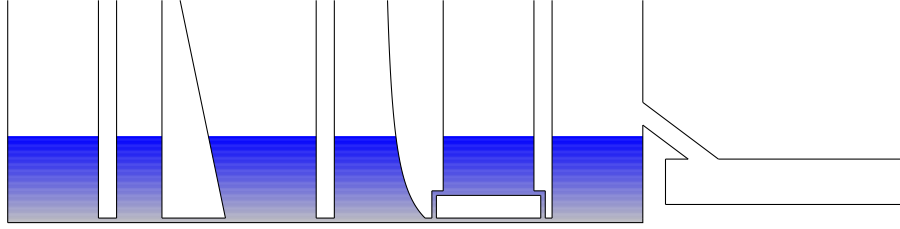
Definition 6 (Nondecreasing Function to $\mathbb{R} \cup \{\text{undefined}\}$). Let $f : \mathbb{R}_{\geq} \rightarrow \mathbb{R} \cup \{\text{undefined}\}$. We say that f is *nondecreasing* if $f|_{f^{-1}(\mathbb{R})}$ is nondecreasing, i.e. if for every $\mu < \mu' \in \mathbb{R}_{\geq}$, if both $f(\mu) \in \mathbb{R}$ and $f(\mu') \in \mathbb{R}$, then $f(\mu) \leq f(\mu')$.

Definition 7 (Communicating-Vessel Equalization). Let $m \in \mathbb{N}$ and let $f_1, \dots, f_m : \mathbb{R}_{\geq} \rightarrow \mathbb{R} \cup \{\text{undefined}\}$ be nondecreasing functions. We define the function $\text{Equalize}_{f_1, \dots, f_m} : \mathbb{R}_{\geq} \rightarrow \mathbb{R} \cup \{\text{undefined}\}$ by

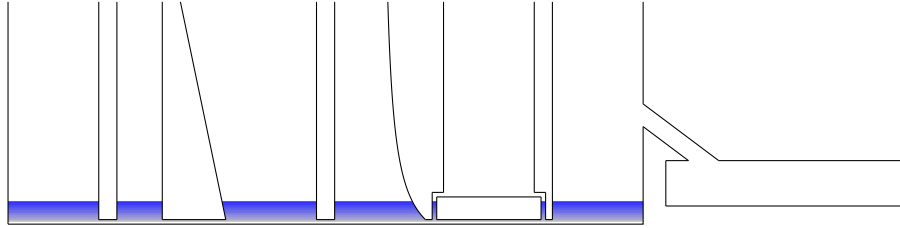
$$\mu \mapsto \begin{cases} f_1(\mu_1) & \exists \mu_1, \dots, \mu_m \in \mathbb{R}_{\geq} : \sum_{j=1}^m \mu_j = \mu \text{ \& \& } f_1(\mu_1) = f_2(\mu_2) = \dots = f_m(\mu_m) \in \mathbb{R} \\ \text{undefined} & \text{otherwise} \end{cases}$$

Remark 2. If $f_1 = f_2 = \dots = f_m$ are defined on all \mathbb{R}_{\geq} , then $\text{Equalize}_{f_1, \dots, f_m}(\mu) = f_1(\frac{\mu}{m})$.

For $f_1, \dots, f_m : \mathbb{R}_{\geq} \rightarrow \mathbb{R}$, one may intuitively think of $\text{Equalize}_{f_1, \dots, f_m}(\mu)$ as exactly the answer to the question raised above: If f_1, \dots, f_m are the functions corresponding (see Fig. 3) to the containers of the first pistons to stop during the process depicted in Fig. 2, and if total amount of liquid in the respective containers when these pistons stop is μ , then $\text{Equalize}_{f_1, \dots, f_m}(\mu)$ is the height in which these pistons stop; $\text{Equalize}_{f_1, \dots, f_m}(\mu) = \text{undefined}$ if it is impossible that all these pistons simultaneously stop when the total amount of liquid in these containers is μ . Alternatively and equivalently, if empty containers corresponding (see Fig. 3) to f_1, \dots, f_m are connected at their base and the resulting system of communicating vessels is jointly filled with μ liquid, then $\text{Equalize}_{f_1, \dots, f_m}(\mu)$ is the resulting liquid surface level; see Fig. 4 for an illustration.



(a) $\text{Equalize}_{f_1, \dots, f_6}(\mu)$ equals the liquid surface level when the containers are jointly filled with μ liquid.



(b) $\text{Equalize}_{f_1, \dots, f_6}(\mu) = \text{undefined}$, as no distribution of μ liquid between the containers results in even liquid surface level across all containers (if the fifth container is empty, then its liquid surface level is defined as the level of its bottom side).

Figure 4: Equalizing the functions from Fig. 3; assume that the connecting tubes are of zero volume.

When two of the functions f_1, \dots, f_m share a plateau height (cf. Corollary 1(2)), then the liquid distribution μ_1, \dots, μ_m may not be well defined; see Fig. 5 for an illustration. Nonetheless, we now show that the resulting surface level $\text{Equalize}_{f_1, \dots, f_m}$ is well defined, i.e. independent of the chosen liquid distribution μ_1, \dots, μ_m .

Lemma 1 (Equalization is Well Defined and Nondecreasing). *Let $m \in \mathbb{N}$ and let $f_1, \dots, f_m : \mathbb{R}_{\geq} \rightarrow \mathbb{R} \cup \{\text{undefined}\}$ be nondecreasing functions. $\text{Equalize}_{f_1, \dots, f_m}$ is a well-defined nondecreasing function.*

The following corollary notes that “connecting a single vessel with itself” has no effect, while connecting $m > 1$ vessels may be done by first connecting subsets of these vessels into “intermediate vessels”, and only then connecting all “intermediate vessels” together; it is for the sake of the latter that we have allowed the functions f_1, \dots, f_m in Definition 7 to assume the value undefined.

Corollary 2. *Let $m \in \mathbb{N}$ and let $f_1, \dots, f_m : \mathbb{R}_{\geq} \rightarrow \mathbb{R} \cup \{\text{undefined}\}$ be nondecreasing functions.*

1. $\text{Equalize}_{f_1} \equiv f_1$.

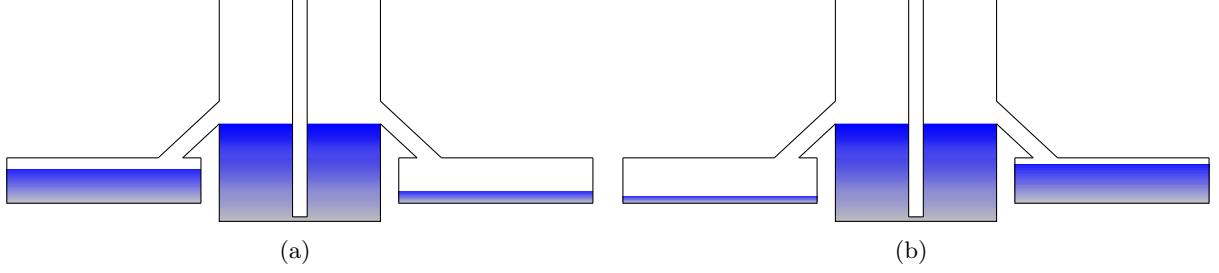


Figure 5: Equalization of two copies of the function from Fig. 3(f), via two distinct liquid distributions. Formally, when $\mu > 2d$, then there exists a continuum of pairs $\mu_1, \mu_2 \in \mathbb{R}_{\geq}$ s.t. $\mu_1 + \mu_2 = \mu$ and $\min\{\mu_1, d\} = \min\{\mu_2, d\}$.

2. $\text{Equalize}_{f_1, \dots, f_m} \equiv \text{Equalize}_{\text{Equalize}_{f_1, \dots, f_{j_1}}, \text{Equalize}_{f_{j_1+1}, \dots, f_{j_2}}, \dots, \text{Equalize}_{f_{j_k+1}, \dots, f_m}},$ for every $k \in [m]$ and $1 \leq j_1 < j_2 < \dots < j_k < m$.

The attentive reader may have noticed that Theorem 1 requires continuity of f_1, \dots, f_n . Indeed, if even one of these functions is discontinuous, then an (even-not-necessarily-strong) Nash equilibrium may not necessarily exist; see Fig. 6 for an illustration. We therefore now

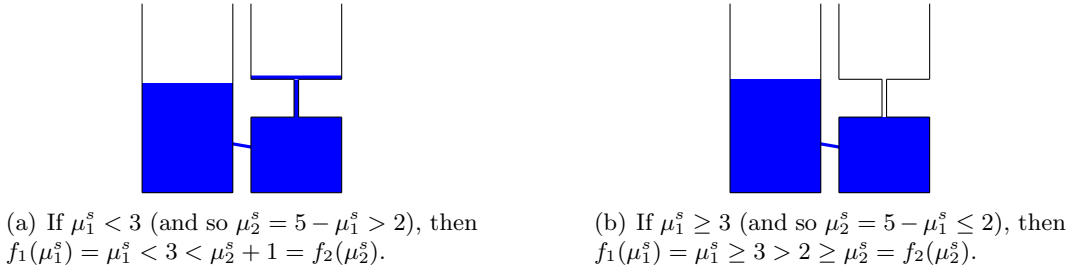


Figure 6: No Nash equilibrium exists when $n = 2$, $f_1 = \text{id}$, $f_2(x) = (x > 2 ? x + 1 : x)$, $\mu^{\{1,2\}} = 5$ and $\mu^{\{1\}} = \mu^{\{2\}} = 0$. (Assume that the tube connecting the two parts of the second vessel is of zero volume.)

turn our attention to equalization of continuous functions.

Definition 8 (Function to $\mathbb{R} \cup \{\text{undefined}\}$: Continuous / Defined on a Suffix of \mathbb{R}_{\geq}). Let $f : \mathbb{R}_{\geq} \rightarrow \mathbb{R} \cup \{\text{undefined}\}$.

- We say that f is *continuous* if $f|_{f^{-1}(\mathbb{R})} : f^{-1}(\mathbb{R}) \rightarrow \mathbb{R}$ is continuous.
- We say that f is *defined on a suffix of \mathbb{R}_{\geq}* if for every $\mu < \mu' \in \mathbb{R}_{\geq}$, if $f(\mu) \in \mathbb{R}$, then $f(\mu') \in \mathbb{R}$ as well.

Lemma 2 (Equalization of Continuous Functions). *Let $m \in \mathbb{N}$ and let $f_1, \dots, f_m : \mathbb{R}_{\geq} \rightarrow \mathbb{R} \cup \{\text{undefined}\}$ be nondecreasing functions.*

1. *If at least one of f_1, \dots, f_m is continuous, then $\text{Equalize}_{f_1, \dots, f_m}$ is continuous.*
2. *If each of f_1, \dots, f_m is continuous and defined on a suffix of \mathbb{R}_{\geq} , then $\text{Equalize}_{f_1, \dots, f_m}$ is continuous and defined on a suffix of \mathbb{R}_{\geq} as well.*

The following corollary shows that for continuous real functions, the only “reason” for their equalization to be undefined is of the type depicted in Fig. 4(b), i.e. an uneven bottom of the corresponding containers.

Corollary 3. Let $m \in \mathbb{N}$ and let $f_1, \dots, f_m : \mathbb{R}_{\geq} \rightarrow \mathbb{R}$ be nondecreasing continuous real functions. $\text{Equalize}_{f_1, \dots, f_m}$ is a real function iff $f_1(0) = f_2(0) = \dots = f_m(0)$.

6.2 Highest-Stopping Pistons and their Height

Following Observation 1 from Section 5, we expect the set of pistons stopping first to be P_G , and expect them to stop at height h_G , for P_G and h_G that we now define using the machinery developed in the previous section.

Definition 9. Let $G = ((f_j)_{j=1}^n; (\mu^R)_{R \in 2_{\neq \emptyset}^{[n]}})$ be a resource selection game. We define:

- $E_G(S) \triangleq \text{Equalize}_{f_k: k \in S} \left(\sum_{R \in 2_{\neq \emptyset}^S} \mu^R \right)$, for every $S \in 2_{\neq \emptyset}^{[n]}$.
- $M_G(S) \triangleq \left\{ S' \in 2_{\neq \emptyset}^S \mid \forall \mu \leq \sum_{R \in 2_{\neq \emptyset}^S \setminus 2_{\neq \emptyset}^{S'}} \mu^R : \text{Equalize}_{f_k: k \in S'}(\mu) \neq E_G(S) \right\}$, for every $S \in 2_{\neq \emptyset}^{[n]}$.
- $D_G \triangleq \{ S \in 2_{\neq \emptyset}^{[n]} \mid E_G(S) \in \mathbb{R} \ \& \ M_G(S) = \emptyset \}$.
- $P_G \triangleq \bigcup_{S \in D_G} \arg \text{Max}_{S \in D_G} E_G(S)$.
- $h_G \triangleq \text{Max}_{S \in D_G} E_G(S)$.

Remark 3. Let $G = ((f_j)_{j=1}^n; (\mu^R)_{R \in 2_{\neq \emptyset}^{[n]}})$ be a resource selection game.

- We show below that in the cases that we study (i.e. when G has a Nash equilibrium or when f_1, \dots, f_n are continuous), P_G is the maximum of $\arg \text{Max}_{S \in D_G} E_G(S)$, and so $h_G = E_G(P_G)$. Furthermore, we show that in these cases $h_G = \text{Max}_{S \in 2_{\neq \emptyset}^{[n]}} E_G(S)$, where the value undefined is treated as $-\infty$ for comparisons by the Max operator, i.e. D_G may be replaced with $2_{\neq \emptyset}^{[n]}$ in the definition of h_G .
- If f_1, \dots, f_n are all (strictly) increasing, then

$$P_G = \bigcup_{S \in 2_{\neq \emptyset}^{[n]}} \arg \text{Max}_{S \in 2_{\neq \emptyset}^{[n]}} E_G(S),$$

where the value undefined is treated as $-\infty$ for comparisons by the Max operator. I.e. D_G may be replaced with $2_{\neq \emptyset}^{[n]}$ in the definition of P_G as well. To see that these two sides in general (i.e. when not all f_1, \dots, f_n are increasing) do not necessarily coincide, consider the game $n = 2$, $f_1 = \text{id}$, $f_2(x) = \min\{x, 2\}$, $\mu^{\{1\}} = 1$, $\mu^{\{2\}} = 3$ and $\mu^{\{1,2\}} = 0$. For this game, $\bigcup \arg \text{Max}_{S \in D_G} E_G(S) = \bigcup \{ \{2\} \} = \{2\}$, albeit $\bigcup \arg \text{Max}_{S \in 2_{\neq \emptyset}^{[n]}} E_G(S) = \bigcup \{ \{2\}, \{1, 2\} \} = \{1, 2\}$. See also Remark 4 below.

Lemma 3. In every resource selection game G , $P_G \neq \emptyset$, and $h_G \in \mathbb{R}$ is well defined.

6.3 Uniqueness and Strength

At the heart of our proof of Theorem 2 lies the Lemma 4, formalizing Observations 1 and 3 to 5 from Section 5. We note that unlike Theorem 1, neither Lemma 4 nor Theorem 2 require the continuity of f_1, \dots, f_n .

Lemma 4 (Uniqueness of Highest-Stopping Pistons and their Height). *Let s be a Nash equilibrium in a resource selection game $G = ((f_j)_{j=1}^n; (\mu^R)_{R \in 2_{\neq \emptyset}^{[n]}})$, and let $P^s \triangleq \arg \text{Max}_{j \in [n]} h_j^s$.*

1. $P^s = P_G$.
2. $h_j^s = h_G$, for every $j \in P^s$.
3. $s_j(R) = 0$ for every $R \in 2_{\neq \emptyset}^{[n]} \setminus 2_{\neq \emptyset}^{P^s}$ and $j \in P^s$.
4. For every $R' \in 2_{\neq \emptyset}^{[n] \setminus P^s}$, define $O(R') \triangleq \{R \in 2_{\neq \emptyset}^{[n]} \mid R \setminus P^s = R'\} \subseteq 2_{\neq \emptyset}^{[n]} \setminus 2_{\neq \emptyset}^{P^s}$. The function $s' : 2_{\neq \emptyset}^{[n] \setminus P^s} \rightarrow \mathbb{R}_{\geq}^{[n] \setminus P^s}$, defined by $s'_j(R') \triangleq \sum_{R \in O(R')} s_j(R)$ for every $j \in [n] \setminus P^s$, constitutes a Nash equilibrium in the game $((f_j)_{j \in [n] \setminus P^s}; (\sum_{R \in O(R')} \mu^R)_{R' \in 2_{\neq \emptyset}^{[n] \setminus P^s}})$. Furthermore, $h_j^s = h_j^{s'}$ for every $j \in [n] \setminus P^s$.

Remark 4. Let $G = ((f_j)_{j=1}^n; (\mu^R)_{R \in 2_{\neq \emptyset}^{[n]}})$ be a resource selection game.

- As mentioned in Remark 3, if f_1, \dots, f_n are all (strictly) increasing, then

$$P_G = \bigcup_{S \in 2_{\neq \emptyset}^{[n]}} \arg \text{Max} E_G(S),$$

where the value undefined is treated as $-\infty$ for comparisons by the Max operator; i.e. D_G may be replaced with $2_{\neq \emptyset}^{[n]}$ in the definition of P_G . To see that Part 1 does not necessarily hold when so defining P_G (when not all f_1, \dots, f_m are increasing), consider once again the game $n = 2$, $f_1 = \text{id}$, $f_2(x) = \min\{x, 2\}$, $\mu^{\{1\}} = 1$, $\mu^{\{2\}} = 3$ and $\mu^{\{1,2\}} = 0$ from Remark 3. In this game, consumption of each player type $\{i\}$ solely from resource i is the unique consumption profile and hence the unique Nash equilibrium — denote it by s . Note that $h_1^s = 1$ and $h_2^s = 2$, and so $P^s = \{2\} = P_G$, albeit $\bigcup \arg \text{Max}_{S \in 2_{\neq \emptyset}^{[n]}} E_G(S) = \bigcup \{\{2\}, \{1, 2\}\} = \{1, 2\}$.

- The r.h.s. of both Lemma 4(1) (both as is and when simplified for increasing functions) and Lemma 4(2), and therefore also the quantifications in Lemma 4(2) and Lemma 4(3), are independent of the choice of s .

The proof of Theorem 2 using Lemma 4 is given in Appendix A.2. This proof effectively follows Algorithm 1 — a succinct algorithm (based upon Lemma 4), which, if any Nash equilibrium exists, directly and explicitly calculates h_j^s for all j in every Nash equilibrium s (without the need to first calculate players' strategies, which are dependent on s).

Full proofs of Corollary 1 and Theorem 3 are given in Appendix A.2. The former is based on Theorem 2 as explained above, and the latter — on the analysis of Lemma 4, following and formalizing an extension of Observation 2 from Section 5. We conclude this section by demonstrating that, as suggested by the manner Theorem 3 is stated, a Nash equilibrium is not necessarily super strong when the condition of Part 2 of this theorem (regarding the plateau heights of f_1, \dots, f_n) is not met. Indeed, consider the resource selection game in which $n = 2$, $f_1 = \text{id}$, $f_2(x) = \min\{x, 3\}$, $\mu^{\{1\}} = 1$, $\mu^{\{2\}} = 2$ and $\mu^{\{1,2\}} = 3$. In this game, a (strong)

Algorithm 1 Direct computation of h_j^s for all $j \in [n]$, regardless of the choice of Nash equilibrium s .

```

1: procedure COMPUTE- $h((f_j)_{j=1}^n; (\mu^R)_{R \in 2_{\neq \emptyset}^{[n]}})$ 
2:   //  $M$  is the set of pistons still in motion.
3:    $M \leftarrow [n]$ 
4:   while  $M \neq \emptyset$  do
5:     // By Lemma 4(4), the pistons  $P$  that stop next are those that stop first in the
6:     //  $|M|$ -resource selection game  $((f_j)_{j \in M}; (\sum_{R \in 2_{\neq \emptyset}^{[n]}: R \cap M = R'} \mu^R)_{R' \in 2_{\neq \emptyset}^M})$ .
7:      $P \leftarrow P((f_j)_{j \in M}; (\sum_{R \in 2_{\neq \emptyset}^{[n]}: R \cap M = R'} \mu^R)_{R' \in 2_{\neq \emptyset}^M})$  // By Lemma 4(1).
8:     //  $h$  is the height at which the pistons  $P$  stop.
9:      $h \leftarrow h((f_j)_{j \in M}; (\sum_{R \in 2_{\neq \emptyset}^{[n]}: R \cap M = R'} \mu^R)_{R' \in 2_{\neq \emptyset}^M})$  // By Lemma 4(2).
10:    for  $j \in P$  do
11:       $h_j \leftarrow h$ 
12:    end for
13:     $M \leftarrow M \setminus P$ 
14:  end while
15:  return  $(h_1, \dots, h_n)$ 
16: end procedure

```

Nash equilibrium is given by $\{1\} \mapsto (1, 0)$, $\{2\} \mapsto (0, 2)$, $\{1, 2\} \mapsto (2, 1)$. Nonetheless, this Nash equilibrium is not super strong, since the coalition of players with types $\{\{1\}, \{1, 2\}\}$ can deviate with $\{1\} \mapsto (1, 0)$ (no change) and $\{1, 2\} \mapsto (0, 3)$, from which players of type $\{1, 2\}$ are unharmed, while players of type $\{1\}$ benefit.

6.4 Existence

We move on to proving the existence of a Nash equilibrium. A full proof of Theorem 1 is given in Appendix A.2. This proof formalizes Observations 1 and 2 from Section 5, effectively following the construction of Fig. 2 and showing that in each step, the pistons stopping are those computed in Algorithm 1. This is done using the following lemma, *constructively* showing, even in the absence of prior knowledge of an existence of a Nash equilibrium, that in each iteration of the algorithm, the liquid that by Lemma 4(3) should be under the pistons P when they stop can be distributed appropriately among them, and that Algorithm 1 indeed finds the sets P in decreasing order of stopping height.

Lemma 5 (P_G, h_G Viable as Highest-Stopping Pistons + Height). *Let $G = ((f_j)_{j=1}^n; (\mu^R)_{R \in 2_{\neq \emptyset}^{[n]}})$ be a resource selection game s.t. f_1, \dots, f_n are continuous.*

1. (Liquid Distribution under P_G). *There exists a consumption profile s in the $|P_G|$ -resource selection game $((f_j)_{j \in P_G}; (\mu^R)_{R \in 2_{\neq \emptyset}^{P_G}})$, s.t. $h_j^s = h_G$ for every $j \in P_G$.*
2. (Pistons Stopping Order). *Let $G' \triangleq ((f_j)_{j \in [n] \setminus P_G}; (\sum_{R \in O(R')} \mu^R)_{R' \in 2_{\neq \emptyset}^{[n] \setminus P_G}})$, where $O(R') \triangleq \{R \in 2_{\neq \emptyset}^{[n]} \mid R \setminus P_G = R'\} \subseteq 2_{\neq \emptyset}^{[n]} \setminus 2_{\neq \emptyset}^{P_G}$ for every $R' \in 2_{\neq \emptyset}^{[n] \setminus P_G}$. If $P_G \neq [n]$, then $h_G > h_{G'}$.*

We once again emphasize that none of the results presented in this paper hinge on any fixed-point theorem, on the Minimax Theorem or any equivalent result, or on the existence of a potential.

7 ID-Dependent Weighting

We briefly describe an example for an extension of the above results; we conjecture that many other extensions exist as well. For $n, k \in \mathbb{N}$, an n -resource/ k -player-type resource selection game with ID-dependent weighting is defined by a treble $((f_j)_{j=1}^n; (f_j^i)_{j \in [n], i \in [k]}; (R^i, \mu^i)_{i=1}^k)$, where $f_j : \mathbb{R}_{\geq} \rightarrow \mathbb{R}$ is a nondecreasing function for every $j \in [n]$, $f_j^i : \mathbb{R}_{\geq} \rightarrow \mathbb{R}_{\geq}$ is an increasing function for every player-type/resource pair $(i, j) \in [k] \times [n]$, and $R^i \in 2_{\neq \emptyset}^{[n]}$ and $\mu^i \in \mathbb{R}_{\geq}$ for every player type $i \in [k]$. For each player type $i \in [k]$, R^i specifies the set of resources from which this player type may consume, while μ^i is the amount to be consumed by all players of this type. As before, for each resource $j \in [n]$, f_j is a function from the consumption amount of this resource to the consumption cost per resource unit. The newly-introduced functions f_j^i indicate the weighting of the consumption of a player i from a resource j (see below).

A *consumption (strategy) profile* in this game is a function $s : [k] \rightarrow \mathbb{R}_{\geq}^{[n]}$ s.t. $s(i) \in \mu^i \cdot \Delta^{R^i}$ for every $i \in [k]$. Given a consumption profile s in this game, we define $\mu_j^s \triangleq \sum_{i=1}^k f_j^i(s_j(i))$ (note the newly-introduced weighting) for every $j \in [n]$ — the weighted load on (i.e. total weighted consumption from) resource j . As before, we define $h_j^s \triangleq f_j(\mu_j^s)$ for every $j \in [n]$. A *Nash equilibrium* in this game is a consumption profile s s.t. for every $i \in [k]$, $h_\ell^s \leq h_j^s$ for every $\ell \in \text{supp}(s(i))$ and $j \in R^i$.

Example 2 (Computing Jobs). Consider a scenario in which the resources are computer servers, and each player wishes to run a large amount of computing jobs, where jobs corresponding to the same player are of a similar nature. Each player $i \in [k]$ may choose between the machines R^i , whose hardware is compatible with player i 's jobs, and would like for all these jobs to complete as soon as possible given this constraint. μ^i in this case is proportional to the amount of jobs of player i , and f_j^i is a linear function s.t. $f_j^i(x)$ is proportional to the number of cycles of machine j required to compute x jobs of player i . (The hardware of each machine may run jobs of some nature more efficiently than jobs of another nature, e.g. machine 2 may run image-processing jobs faster than text-analysis ones, while machine 3 may run the latter faster than the former.) For each $j \in [n]$, we choose f_j s.t. $h_j^s = f_j(\mu_j^s)$ is proportional to the number of seconds required for μ_j^s cycles to complete. (Assume that the resources of each machine are parallelized between its different users, so that their jobs all complete at the same time.)

We note that by setting $f_j^i = \text{id}$ for all i, j , we obtain a resource selection game as in the previous sections, and so this is a strict generalization of resource selection games as defined there. Intuitively, the construction from Fig. 2 may be adapted to this generalized framework by inserting “compressors/expanders” into the tubes between balloons corresponding to the same player type. E.g. if $\mu^1 = \{1, 2\}$, $f_1^1(x) = x$ and $f_2^1(x) = 2x$, then the balloon system corresponding to player type 1 consists of two balloons — one in container 1 and the other in container 2, connected by a compressor/expander tube s.t. for each drop of liquid that enters the tube from the balloon in container 1, two drops exit into the balloon in container 2, and for every two drops of liquid that enter the tube from the balloon in container 2, one drop exits into the balloon in container 1.

The first thing that we note about this generalized game is that it no longer holds that h_j^s is independent of the choice of a Nash equilibrium s ; see Fig. 7 for an illustration. Nonetheless, if we accept the physical intuition that when compressed via pistons, any of the liquid distributions given in Fig. 7 eventually reaches that in Fig. 7(d), then our construction can be shown to yield a strong (and under the conditions of Theorem 3(2), super-strong) Nash equilibrium, and uniqueness of h_j^s can still be shown to hold among strong Nash equilibria. Formally, Theorem 3(1) no longer holds w.r.t. the game $((f_j)_{j=1}^n; (f_j^i)_{j \in [n], i \in [k]}; (R^i, \mu^i)_{i=1}^k)$, while Theorems 1, 2 and 3(2) and

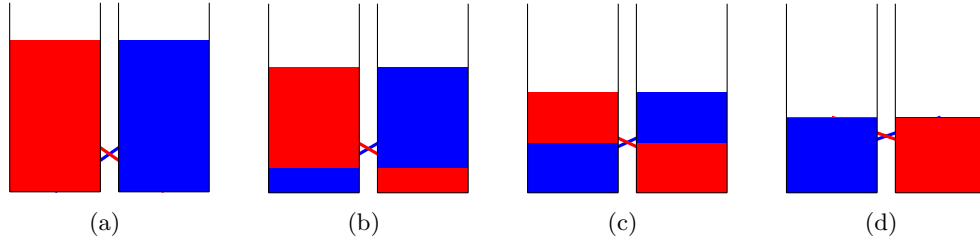


Figure 7: Liquid distributions among balloons, corresponding to a plethora of Nash equilibria s with distinct h_j^s , when $n = 2$, $k = 2$ (blue corresponding to $i = 1$, and red — to $i = 2$), $f_1 = f_2 = \text{id}$, $\mu^1 = \mu^2 = 1$, $f_1^1(x) = f_2^2(x) = x$ and $f_2^1(x) = f_1^2(x) = 2x$. Only the Nash equilibrium depicted in Fig. 7(d) is strong (in fact, it is super strong).

Corollary 1 still hold w.r.t. this game when replacing every occurrence of “Nash” with “strong Nash”. The formal analysis leading to these results is conceptually similar to that presented in Section 6.

8 Further Research

The results of this paper and of Gonczarowski and Tennenholtz (2014), as well as the earlier results of Kaminsky (2000) and of Fisher, show not only that physical hydraulic systems may be a fruitful source of intuition for proofs regarding equilibria, but furthermore that they may be used to naturally “calculate” a variety of flavours of equilibria. It would be interesting to rigorously define a “hydraulic” calculation, and to study its strength and limitations.

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A Proofs and Auxiliary Results

A.1 Proofs of Lemmas and Corollaries from Section 6, and Auxiliary Results

A.1.1 Communicating-Vessel Equalization

We begin with an immediate consequence of Definition 6.

Lemma 6. *Let $m \in \mathbb{N}$ and let $f_1, \dots, f_m : \mathbb{R}_{\geq} \rightarrow \mathbb{R} \cup \{\text{undefined}\}$ be nondecreasing functions. Let $\mu_1, \dots, \mu_m \in \mathbb{R}_{\geq}$ s.t. $f_1(\mu_1) = f_2(\mu_2) = \dots = f_m(\mu_m) \in \mathbb{R}$ and let $\mu'_1, \dots, \mu'_m \in \mathbb{R}_{\geq}$ s.t. $\sum_{j=1}^m \mu'_j \geq \sum_{j=1}^m \mu_j$.*

1. *If $f_1(\mu'_1), f_2(\mu'_2), \dots, f_m(\mu'_m) \in \mathbb{R}$, then there exists $j \in [m]$ s.t. $f_j(\mu'_j) \geq f_j(\mu_j)$.*
2. *If $f_1(\mu'_1) = f_2(\mu'_2) = \dots = f_m(\mu'_m) \in \mathbb{R}$, then $f_1(\mu'_1) \geq f_1(\mu_1)$.*

Proof. For Part 1, since $\sum_{j=1}^m \mu'_j \geq \sum_{j=1}^m \mu_j$, there exists $j \in [m]$ s.t. $\mu'_j \geq \mu_j$. As f_j is nondecreasing and as $\mu_j, \mu'_j \in f_j^{-1}(\mathbb{R})$, we have $f_j(\mu'_j) \geq f_j(\mu_j)$, as required. For Part 2, by Part 1 there exists $j \in [m]$ s.t. $f_j(\mu'_j) \geq f_j(\mu_j)$; therefore, $f_1(\mu'_1) = f_j(\mu'_j) \geq f_j(\mu_j) = f_1(\mu_1)$. \square

Proof of Lemma 1. We start by showing that $\text{Equalize}_{f_1, \dots, f_m}(\mu)$ is well defined for every $\mu \in \mathbb{R}_{\geq}$. We have to show that if there exist $\mu_1, \dots, \mu_m \in \mathbb{R}_{\geq}$ s.t. $\sum_{j=1}^m \mu_m = \mu$ and $f_1(\mu_1) = f_2(\mu_2) = \dots = f_m(\mu_m) \in \mathbb{R}$, then $f_1(\mu'_1) = f_1(\mu_1)$ for every $\mu'_1, \dots, \mu'_m \in \mathbb{R}_{\geq}$ s.t. $\sum_{j=1}^m \mu'_m = \mu$ and $f_1(\mu'_1) = f_2(\mu'_2) = \dots = f_m(\mu'_m) \in \mathbb{R}$ as well. This follows directly from Lemma 6(2), as both $\sum_{j=1}^m \mu_m \leq \sum_{j=1}^m \mu'_m$ and $\sum_{j=1}^m \mu'_m \leq \sum_{j=1}^m \mu_m$.

The fact that $\text{Equalize}_{f_1, \dots, f_m}$ is nondecreasing follows directly from Lemma 6(2) as well. \square

Proof of Corollary 2. Part 1 follows directly by definition, as when $m = 1$, we always have $\mu_1 = \mu$. We move on to proving Part 2; Let $k \in [m]$ and $1 \leq j_1 < j_2 < \dots < j_k < m$; define $j_0 \triangleq 0$ and $j_{k+1} \triangleq m$.

If $h \triangleq \text{Equalize}_{f_1, \dots, f_m}(\mu) \in \mathbb{R}$, then there exist $\mu_1, \dots, \mu_m \in \mathbb{R}_{\geq}$ s.t. $\sum_{j=1}^m \mu_m = \mu$ and $f_1(\mu_1) = f_2(\mu_2) = \dots = f_m(\mu_m) = h$. Let $i \in [k+1]$; as $f_{j_{i-1}+1}(\mu_{j_{i-1}+1}) = f_{j_{i-1}+2}(\mu_{j_{i-1}+2}) = \dots = f_{j_i}(\mu_{j_i})$, we have that $\text{Equalize}_{f_{j_{i-1}+1}, \dots, f_{j_i}}(\sum_{\ell=j_{i-1}+1}^{j_i} \mu_{\ell}) = f_{j_{i-1}+1}(\mu_{j_{i-1}+1}) = h$. Hence, we have $h = \text{Equalize}_{\text{Equalize}_{f_1, \dots, f_{j_1}}, \text{Equalize}_{f_{j_1+1}, \dots, f_{j_2}}, \dots, \text{Equalize}_{f_{j_k+1}, \dots, f_m}}(\sum_{i=1}^{k+1} \sum_{\ell=j_{i-1}+1}^{j_i} \mu_{\ell}) = \text{Equalize}_{\text{Equalize}_{f_1, \dots, f_{j_1}}, \text{Equalize}_{f_{j_1+1}, \dots, f_{j_2}}, \dots, \text{Equalize}_{f_{j_k+1}, \dots, f_m}}(\mu)$, as required.

Conversely, if $h \triangleq \text{Equalize}_{\text{Equalize}_{f_1, \dots, f_{j_1}}, \text{Equalize}_{f_{j_1+1}, \dots, f_{j_2}}, \dots, \text{Equalize}_{f_{j_k+1}, \dots, f_m}}(\mu) \in \mathbb{R}$, then there exist $\tilde{\mu}_1, \dots, \tilde{\mu}_{k+1}$ s.t. $\sum_{i=1}^{k+1} \tilde{\mu}_i = \mu$ and $\text{Equalize}_{f_{j_{i-1}+1}, \dots, f_{j_i}}(\tilde{\mu}_i) = h$ for every $i \in [k+1]$. Therefore, for every $i \in [k+1]$, there exist $\mu_{j_{i-1}+1}, \dots, \mu_{j_i}$ s.t. $\sum_{\ell=j_{i-1}+1}^{j_i} \mu_{\ell} = \tilde{\mu}_i$ and $f_{j_{i-1}+1}(\mu_{j_{i-1}+1}) = \dots = f_{j_i}(\mu_{j_i}) = h$. As $\sum_{j=1}^m \mu_m = \sum_{i=1}^{k+1} \tilde{\mu}_i = \mu$ and $h = f_1(\mu_1) = f_2(\mu_2) = \dots = f_m(\mu_m)$, we have that $\text{Equalize}_{f_1, \dots, f_m}(\mu) = h$, as required. \square

Proof of Lemma 2. By Corollary 2, when proving either part it is enough to consider the case in which $m = 2$. (The case $m = 1$ follows from Corollary 2(1), while the case $m > 2$ follows from the case $m = 2$ by iteratively applying Corollary 2(2).)

We start by proving Part 1. Let $\mu \in \mathbb{R}_{\geq}$ s.t. $h \triangleq \text{Equalize}_{f_1, f_2}(\mu) \in \mathbb{R}$ and let $\varepsilon > 0$; assume w.l.o.g. that f_1 is continuous. By definition of h , there exists $\mu_1 \in [0, \mu]$ s.t. $f_1(\mu_1) = f_2(\mu - \mu_1) = h$. By continuity of f_1 , there exists $\delta > 0$ s.t. $|f_1(\mu') - h| < \varepsilon$ for every $\mu' \in (\mu - \delta, \mu + \delta) \cap f_1^{-1}(\mathbb{R})$. Let $\mu' \in (\mu - \delta, \mu + \delta) \cap \text{Equalize}_{f_1, f_2}^{-1}(\mathbb{R})$; by definition, there exists $\mu'_1 \in [0, \mu']$ s.t. $f_1(\mu'_1) = f_2(\mu' - \mu'_1) = h' \triangleq \text{Equalize}_{f_1, f_2}(\mu')$. If $h' = h$, then we trivially have $|h' - h| = 0 < \varepsilon$, as required; assume, therefore, that $h' \neq h$. We show that $\mu'_1 \in (\mu_1 - \delta, \mu_1 + \delta)$ by considering

two cases. If $h' > h$, then as f_1, f_2 are nondecreasing and as $f_1(\mu_1) = h < h' = f_1(\mu'_1)$ and $f_2(\mu - \mu_1) = h < h' = f_1(\mu' - \mu'_1)$, we have $\mu_1 < \mu'_1$ and $\mu - \mu_1 < \mu' - \mu'_1$; combining these, we have that $\mu'_1 \in (\mu_1, \mu_1 + \mu' - \mu) \subseteq (\mu_1, \mu_1 + \delta) \subseteq (\mu_1 - \delta, \mu_1 + \delta)$ in this case. If $h' < h$, then similarly, as f_1, f_2 are nondecreasing and as $f_1(\mu_1) = h > h' = f_1(\mu'_1)$ and $f_2(\mu - \mu_1) = h > h' = f_1(\mu' - \mu'_1)$, we have $\mu_1 > \mu'_1$ and $\mu - \mu_1 > \mu' - \mu'_1$; combining these, we have that $\mu'_1 \in (\mu_1 + \mu' - \mu, \mu_1) \subseteq (\mu_1 - \delta, \mu_1) \subseteq (\mu_1 - \delta, \mu_1 + \delta)$ in this case as well. By definition of δ and as $f_1(\mu') = h' \in \mathbb{R}$, we obtain $|h' - h| = |f_1(\mu') - h| < \varepsilon$, as required.

We move on to proving Part 2. By Part 1, $\text{Equalize}_{f_1, f_2}$ is continuous; it is therefore left to show that $\text{Equalize}_{f_1, f_2}$ is defined on a suffix of \mathbb{R}_{\geq} . Recall that for every $\mu \in \mathbb{R}$, by definition $\text{Equalize}_{f_1, f_2}(\mu) \in \mathbb{R}$ iff there exists $\mu_1 \in [0, \mu]$ s.t. $f_1(\mu_1) = f_2(\mu - \mu_1) \in \mathbb{R}$. Let $\mu \in \mathbb{R}_{\geq}$ s.t. $\text{Equalize}_{f_1, f_2}(\mu) \in \mathbb{R}$; therefore, there exists $\mu_1 \in [0, \mu]$ s.t. $f_1(\mu_1) = f_2(\mu - \mu_1) \in \mathbb{R}$. Let $\mu' > \mu$; note that as $\mu_1 \leq \mu$, we have $\mu_1 + \mu' - \mu \leq \mu'$. Since $f_1(\mu_1), f_2(\mu - \mu_1) \in \mathbb{R}$ and as $\mu' > \mu$, we have by f_1 and f_2 being defined on a suffix of \mathbb{R}_{\geq} , that $f_1(\mu_1 + \mu' - \mu), f_2(\mu' - \mu_1) \in \mathbb{R}$ as well. Furthermore, as f_1 and f_2 are nondecreasing, we have $f_1(\mu_1) = f_2(\mu - \mu_1) \leq f_2(\mu' - \mu_1)$ and $f_1(\mu_1 + \mu' - \mu) \geq f_1(\mu_1) = f_2(\mu - \mu_1) = f_2(\mu' - (\mu_1 + \mu' - \mu))$. By continuity of f_1 and f_2 and as $[\mu_1, \mu_1 + \mu' - \mu] \subseteq f_1^{-1}(\mathbb{R})$ and $[\mu' - (\mu_1 + \mu' - \mu), \mu' - \mu_1] = [\mu - \mu_1, \mu' - \mu_1] \subseteq f_2^{-1}(\mathbb{R})$, we have by the intermediate value theorem that there exists $\mu'_1 \in [\mu_1, \mu_1 + \mu' - \mu] \subseteq [0, \mu']$ s.t. $f_1(\mu'_1) = f_2(\mu' - \mu'_1) \in \mathbb{R}$, as required. \square

Proof of Corollary 3. By Lemma 2(2), $\text{Equalize}_{f_1, \dots, f_m}$ is a real function iff $\text{Equalize}_{f_1, \dots, f_m}(0) \in \mathbb{R}$, which by definition holds iff $f_1(0) = f_2(0) = \dots = f_m(0)$. \square

A.1.2 Highest-Stopping Pistons and their Height

Proof of Lemma 3. By definition, $S \notin M_G(S)$ for every $S \in 2_{\neq \emptyset}^{[n]}$ (by taking $\mu \triangleq \sum_{R \in 2_{\neq \emptyset}^S} \mu^R$). Therefore, $M_G(\{1\}) = \emptyset$; furthermore, by Corollary 2(1), $E_G(\{1\}) = \text{Equalize}_{f_1}(\mu^{\{1\}}) = f_1(\mu^{\{1\}}) \in \mathbb{R}$. Therefore, $\{1\} \in D_G$. In particular, we have that $D_G \neq \emptyset$, and so, by finiteness of D_G , we have that $P_G \neq \emptyset$ and that $h_G \in \mathbb{R}$ is well defined. \square

A.1.3 Uniqueness and Strength

Proof of Lemma 4. We start by proving Part 3. Let $R \in 2_{\neq \emptyset}^{[n]}$ s.t. there exists $j \in P^s$ s.t. $s_j(R) > 0$; it is enough to show that $R \in 2_{\neq \emptyset}^{P^s}$. By definition of s , $h_j^s \leq h_k^s$ for every $k \in R$, and as $h_j^s = \max_{i \in [n]} h_i^s \geq h_k^s$, we have $h_k^s = h_j^s$ and so $k \in P^s$ for every $k \in R$. Therefore, $R \in 2_{\neq \emptyset}^{P^s}$ as required.

We move on to prove Part 4. We first show that s' is a consumption profile in the game $G' \triangleq ((f_j)_{j \in [n] \setminus P^s}; (\sum_{R \in O(R')} \mu^R)_{R' \in 2_{\neq \emptyset}^{[n] \setminus P^s}})$. Let $R' \in 2_{\neq \emptyset}^{[n] \setminus P^s}$. By definition of s' , we have that $s'_j(R') = \sum_{R \in O(R')} s_j(R) \geq 0$ for every $j \in [n] \setminus P^s$; furthermore, for every $j \in ([n] \setminus P^s) \setminus R'$, we have by definition that $j \notin R$ for every $R \in O(R')$, and so $s'_j(R') = \sum_{R \in O(R')} s_j(R) = 0$. Finally, we have that $\sum_{j \in [n] \setminus P^s} s'_j(R') = \sum_{j \in [n] \setminus P^s} \sum_{R \in O(R')} s_j(R) = \sum_{R \in O(R')} \sum_{j \in [n] \setminus P^s} s_j(R) = \sum_{R \in O(R')} \sum_{j \in [n]} s_j(R) = \sum_{R \in O(R')} \mu^R$, where the penultimate equality is by Part 3.

We move on to show that $h_j^s = h_j^{s'}$ for every $j \in [n] \setminus P^s$. By definition of s' , we have for every $j \in [n] \setminus P^s$ that $\mu_j^{s'} = \sum_{R' \in 2_{\neq \emptyset}^{[n] \setminus P^s}} s'_j(R') = \sum_{R' \in 2_{\neq \emptyset}^{[n] \setminus P^s}} \sum_{R \in O(R')} s_j(R) = \sum_{R \in 2_{\neq \emptyset}^{[n] \setminus P^s}} s_j(R) = \sum_{R \in 2_{\neq \emptyset}^{[n]}} s_j(R) = \mu_j^s$ (where the penultimate equality is since $j \notin R$ for every $R \in 2_{\neq \emptyset}^{P^s}$), and hence $h_j^{s'} = f_j(\mu_j^{s'}) = f_j(\mu_j^s) = h_j^s$, as required.

We conclude by showing s' is indeed a Nash equilibrium in G' . Let $R' \in 2_{\neq \emptyset}^{[n] \setminus P^s}$, and let $k \in \text{supp}(s'(R'))$ and $j \in R'$. As $0 < s'_k(R') = \sum_{R \in O(R')} s_k(R)$, we have that there exists

$R \in O(R')$ s.t. $k \in \text{supp}(s(R))$. As $j \in R' \subseteq R$, since s' is a Nash equilibrium in G , we have that $h_k^s \leq h_j^s$; therefore, $h_k^{s'} = h_k^s \leq h_j^s = h_j^{s'}$ and so s' is a Nash equilibrium in G' , as required.

Before moving on to prove Parts 1 and 2, we prove a few auxiliary results. We first show that

$$\forall j \in P^s : h_j^s = E_G(P^s) = \text{Equalize}_{f_k: k \in P^s} \left(\sum_{R \in 2_{\neq \emptyset}^{P^s}} \mu^R \right). \quad (1)$$

By definition of P^s , $f_j(\mu_j^s) = h_j^s = h_k^s = f_k(\mu_k^s)$ for every $j, k \in P^s$. Therefore, $h_j^s = \text{Equalize}_{f_k: k \in P^s} (\sum_{k \in P^s} \mu_k^s)$ for every $j \in P^s$. It is therefore enough to show that $\sum_{k \in P^s} \mu_k^s = \sum_{R \in 2_{\neq \emptyset}^{P^s}} \mu^R$. Indeed, we have $\sum_{k \in P^s} \mu_k^s = \sum_{k \in P^s} \sum_{R \in 2_{\neq \emptyset}^{[n]}} s_k(R) = \sum_{R \in 2_{\neq \emptyset}^{[n]}} \sum_{k \in P^s} s_k(R) = \sum_{R \in 2_{\neq \emptyset}^{P^s}} \sum_{k \in P^s} s_k(R) = \sum_{R \in 2_{\neq \emptyset}^{P^s}} \mu^R$, where the penultimate equality is by Part 3, and the last equality is since $s(R) \in \mu^R \cdot \Delta^R \subseteq \mu^R \cdot \Delta^{P^s}$ for every $R \in 2_{\neq \emptyset}^{P^s}$.

Next, we show that for every $S \in 2_{\neq \emptyset}^{[n]}$ s.t. $E_G(S) = \text{Equalize}_{f_k: k \in S} (\sum_{R \in 2_{\neq \emptyset}^S} \mu^R) \in \mathbb{R}$, there exists $k \in S$ s.t. $f_k(\mu_k^s) \geq E_G(S)$. Indeed, since $s(R) \in \mu^R \cdot \Delta^R \subseteq \mu^R \cdot \Delta^S$ for every $R \in 2_{\neq \emptyset}^S$, we have that $\sum_{R \in 2_{\neq \emptyset}^S} \mu^R = \sum_{R \in 2_{\neq \emptyset}^S} \sum_{k \in S} s_k(R) \leq \sum_{R \in 2_{\neq \emptyset}^{[n]}} \sum_{k \in S} s_k(R) = \sum_{k \in S} \sum_{R \in 2_{\neq \emptyset}^{[n]}} s_k(R) = \sum_{k \in S} \mu_k^s$ and so, by Lemma 6(1), there exists $k \in S$ s.t. $f_k(\mu_k^s) \geq \text{Equalize}_{f_k: k \in S} (\sum_{R \in 2_{\neq \emptyset}^S} \mu^R) = E_G(S)$, as required.

We now show that $P^s \in \arg \text{Max}_{S \in 2_{\neq \emptyset}^{[n]}} E_G(S)$, where the value undefined is treated as $-\infty$ for comparisons by the Max operator. Let $S \in 2_{\neq \emptyset}^{[n]}$ s.t. $E_G(S) \in \mathbb{R}$. As shown above, there exists $k \in S$ s.t. $f_k(\mu_k^s) \geq E_G(S)$. Therefore, by Eq. (1) and by definition of P^s we obtain that $E_G(P^s) = \text{Max}_{j \in [n]} h_j^s \geq h_k^s = f_k(\mu_k^s) \geq E_G(S)$, and so indeed $P^s \in \arg \text{Max}_{S \in 2_{\neq \emptyset}^{[n]}} E_G(S)$.

Finally, we show that $M_G(P^s) = \emptyset$. We have to show that for every $S \in 2_{\neq \emptyset}^{P^s}$ there exists $\mu \leq \sum_{R \in 2_{\neq \emptyset}^{P^s} \setminus 2_{\neq \emptyset}^{P^s \setminus S}} \mu^R$ s.t. $\text{Equalize}_{f_k: k \in S}(\mu) = E_G(P^s)$. Let, therefore, $S \in 2_{\neq \emptyset}^{P^s}$ and define $\mu \triangleq \sum_{j \in S} \mu_j^s$. By Eq. (1) and by definition of P^s , it is enough to show that both $\text{Equalize}_{f_k: k \in S}(\mu) = \text{Max}_{j \in [n]} h_j^s$ and $\mu \leq \sum_{R \in 2_{\neq \emptyset}^{P^s} \setminus 2_{\neq \emptyset}^{P^s \setminus S}} \mu^R$. Since $S \subseteq P^s$, we have $f_k(\mu_k^s) = h_k^s = \text{Max}_{j \in [n]} h_j^s$ for every $k \in S$, and so, by definition, $\text{Equalize}_{f_k: k \in S}(\mu) = \text{Equalize}_{f_k: k \in S} (\sum_{j \in S} \mu_j^s) = \text{Max}_{j \in [n]} h_j^s$. For every $j \in S$, we have $\mu_j^s = \sum_{R \in 2_{\neq \emptyset}^{[n]}} s_j(R) = \sum_{R \in 2_{\neq \emptyset}^{P^s}} s_j(R) = \sum_{R \in 2_{\neq \emptyset}^{P^s} \setminus 2_{\neq \emptyset}^{P^s \setminus S}} s_j(R)$, where the penultimate equality is by Part 3 since $j \in S \subseteq P^s$, and the last inequality is since $j \notin R$ for every $R \in 2_{\neq \emptyset}^{P^s \setminus S}$. Therefore, $\sum_{j \in S} \mu_j^s = \sum_{j \in S} \sum_{R \in 2_{\neq \emptyset}^{P^s} \setminus 2_{\neq \emptyset}^{P^s \setminus S}} s_j(R) = \sum_{R \in 2_{\neq \emptyset}^{P^s} \setminus 2_{\neq \emptyset}^{P^s \setminus S}} \sum_{j \in S} s_j(R) \leq \sum_{R \in 2_{\neq \emptyset}^{P^s} \setminus 2_{\neq \emptyset}^{P^s \setminus S}} \sum_{j \in [n]} s_j(R) = \sum_{R \in 2_{\neq \emptyset}^{P^s} \setminus 2_{\neq \emptyset}^{P^s \setminus S}} \mu^R$, as required, and so $M_G(P^s) = \emptyset$.

We now move on to proving Part 1; we do so by showing mutual containment between the two sides of the equality.

\subseteq : It is enough to show that $P^s \in \arg \text{Max}_{S \in D_G} E_G(S)$. As $M_G(P^s) = \emptyset$ and as by Eq. (1) $E_G(P^s) \in \mathbb{R}$, we have $P^s \in D_G$. As $E_G(P^s) = \text{Max}_{S \in 2_{\neq \emptyset}^{[n]}} E_G(S) \geq \text{Max}_{S \in D_G} E_G(S)$, we therefore have $P^s \in \arg \text{Max}_{S \in D_G} E_G(S)$, as required.

\supseteq : We must show that $S \subseteq P^s$ for every $S \in \arg \text{Max}_{S'' \in D_G} E_G(S'')$. Define $S' \triangleq S \setminus P^s \in 2^S$ and assume for contradiction that $S' \neq \emptyset$. It is enough to show that $\text{Equalize}_{f_k: k \in S'}(\mu) \neq E_G(S)$ for every $\mu \leq \sum_{R \in 2_{\neq \emptyset}^S \setminus 2_{\neq \emptyset}^{S \setminus S'}} \mu^R$, since this implies $S' \in M_G(S)$ — a contradiction, as $S \in D_G$. Let, therefore, $\mu \leq \sum_{R \in 2_{\neq \emptyset}^S \setminus 2_{\neq \emptyset}^{S \setminus S'}} \mu^R$; as by Lemma 3, $E_G(S) = h_G \in \mathbb{R}$, it is enough to show that if $\text{Equalize}_{f_k: k \in S'}(\mu) \in \mathbb{R}$, then $\text{Equalize}_{f_k: k \in S'}(\mu) < E_G(S)$. Recall from the proof of the other direction (" \subseteq ") that $P^s \in \arg \text{Max}_{S'' \in D_G} E_G(S'')$; therefore, by definition of S , by Eq. (1) and by definition of P^s , we obtain that $E_G(S) = E_G(P^s) = \text{Max}_{k \in [n]} h_k^s$. It is thus enough to

show that $\text{Equalize}_{f_k:k \in S'}(\mu) < \text{Max}_{k \in [n]} h_k^s$.

By definition of S' and P^s , we have that $h_j^s < \text{Max}_{k \in [n]} h_k^s$ for every $j \in S'$ and $h_j^s = \text{Max}_{k \in [n]} h_k^s$ for every $j \in S \setminus S'$; ergo, $s_j(R) = 0$ for every $j \in S \setminus S'$ and $R \in 2_{\neq \emptyset}^S \setminus 2_{\neq \emptyset}^{S \setminus S'}$. Hence, $\sum_{j \in S'} \mu_j^s = \sum_{j \in S'} \sum_{R \in 2_{\neq \emptyset}^{[n]}} s_j(R) \geq \sum_{j \in S'} \sum_{R \in 2_{\neq \emptyset}^S \setminus 2_{\neq \emptyset}^{S \setminus S'}} s_j(R) = \sum_{R \in 2_{\neq \emptyset}^S \setminus 2_{\neq \emptyset}^{S \setminus S'}} \sum_{j \in S'} s_j(R) = \sum_{R \in 2_{\neq \emptyset}^S \setminus 2_{\neq \emptyset}^{S \setminus S'}} \sum_{j \in S} s_j(R) = \sum_{R \in 2_{\neq \emptyset}^S \setminus 2_{\neq \emptyset}^{S \setminus S'}} \mu^R \geq \mu$. Therefore, by Lemma 6(1) there exists $j \in S'$ s.t. $f_j(\mu_j^s) \geq \text{Equalize}_{f_k:k \in S'}(\mu)$, and thus $\text{Equalize}_{f_k:k \in S'}(\mu) \leq f_j(\mu_j^s) = h_j^s < \text{Max}_{k \in [n]} h_k^s$, as required.

We conclude by proving Part 2. Recall from the proof of the first direction (\subseteq) of Part 1 that $E_G(P^s) = \text{Max}_{S \in D_G} E_G(S)$. Therefore, by Eq. (1), $h_j^s = E_G(P^s) = \text{Max}_{S \in D_G} E_G(S) = h_G$ for every $j \in P^s$, as required. \square

A.1.4 Constrained Distribution

Before moving on to prove Lemma 5, we first formulate and prove a combinatorial result that we use in the proof of this lemma.

Definition 10 (Distribution Constraint).

1. A *distribution constraint* is a pair $((\mu^R)_{R \in 2_{\neq \emptyset}^{[n]}}, ([t_j, T_j])_{j \in [n]})$, where $n \in \mathbb{N}$, $\mu^R \in \mathbb{R}_{\geq}$ for every $R \in 2_{\neq \emptyset}^{[n]}$, and $t_j \leq T_j \in \mathbb{R}_{\geq}$ for every $j \in [n]$.
2. We say that a distribution constraint $C = ((\mu^R)_{R \in 2_{\neq \emptyset}^{[n]}}, ([t_j, T_j])_{j \in [n]})$ is *satisfiable* if there exist $(\mu_j^R)_{j \in [n]}^{R \in 2_{\neq \emptyset}^{[n]}}$ s.t. $(\mu_j^R)_{j \in [n]} \in \mu^R \cdot \Delta^R$ for every $R \in 2_{\neq \emptyset}^{[n]}$ and $\sum_{R \in 2_{\neq \emptyset}^{[n]}} \mu_j^R \in [t_j, T_j]$ for every $j \in [n]$.
3. Given a distribution constraint $C = ((\mu^R)_{R \in 2_{\neq \emptyset}^{[n]}}, ([t_j, T_j])_{j \in [n]})$, for every $S \in 2_{\neq \emptyset}^{[n]}$ we define $m_C(S) \triangleq \sum_{R \in 2_{\neq \emptyset}^S} \mu^R$, $M_C(S) \triangleq \sum_{R \in 2_{\neq \emptyset}^{[n]} \setminus 2_{\neq \emptyset}^{[n] \setminus S}} \mu^R$, $t_C(S) \triangleq \sum_{j \in S} t_j$ and $T_C(S) \triangleq \sum_{j \in S} T_j$. We say that C is *normal* if both $t_C(S) \leq M_C(S)$ and $m_C(S) \leq T_C(S)$ for every $S \in 2_{\neq \emptyset}^{[n]}$.

We note that it is trivial to show that every satisfiable distribution constraint is normal. In this section, we *constructively* show (without the use of e.g. linear programming) that the other direction holds as well, and give a procedure for explicitly finding a solution to (i.e. a witness to the satisfiability of) any given normal distribution:

Lemma 7. *Every normal distribution constraint is satisfiable.*

Before proving Lemma 7, we first develop some machinery.

Lemma 8. *Let $C = ((\mu^R)_{R \in 2_{\neq \emptyset}^{[n]}}, ([t_j, T_j])_{j \in [n]})$ be a normal distribution constraint.*

1. $M_C(S \cup S') = t_C(S \cup S')$, for every $S, S' \in 2_{\neq \emptyset}^{[n]}$ s.t. $M_C(S) = t_C(S)$ and $M_C(S') = t_C(S')$.
2. $m_C(S \cap S') = T_C(S \cap S')$, for every $S, S' \in 2_{\neq \emptyset}^{[n]}$ s.t. $m_C(S) = T_C(S)$, $m_C(S') = T_C(S')$ and $S \cap S' \neq \emptyset$.

Proof. For every $S, S' \in 2_{\neq \emptyset}^{[n]}$ s.t. $M_C(S) = t_C(S)$ and $M_C(S') = t_C(S')$, we have $M_C(S \cup S') \leq M_C(S) + M_C(S') - M_C(S \cap S') = t_C(S) + t_C(S') - M_C(S \cap S') \leq t_C(S) + t_C(S') - t_C(S \cap S') = t_C(S \cup S')$, as required. (The other side of the inequality follows from normality of C .)

For every $S, S' \in 2_{\neq \emptyset}^{[n]}$ s.t. $m_C(S) = T_C(S)$, $m_C(S') = T_C(S')$ and $S \cap S' \neq \emptyset$, we have $m_C(S \cap S') \geq m_C(S) + m_C(S') - m_C(S \cup S') = T_C(S) + T_C(S') - m_C(S \cup S') \geq T_C(S) + T_C(S') - T_C(S \cup S') = T_C(S \cap S')$, as required. (Once again, the other side of the inequality follows from normality of C .) \square

Lemma 9 (Moving Mass from R to $\{n\}$). *Let $C = ((\mu^R)_{R \in 2_{\neq \emptyset}^{[n]}}, ([t_j, T_j])_{j \in [n]})$ be a normal distribution constraint. For every $R \in 2_{\neq \emptyset}^{[n]}$ s.t. $\{n\} \subsetneq R$, we define*

$$Q_C^R \triangleq \min \left\{ \min_{\substack{S \in 2_{\neq \emptyset}^{[n-1]}: \\ S \cap R \neq \emptyset}} (M_C(S) - t_C(S)), \min_{\substack{S \in 2_{\neq \emptyset}^{[n]}: \\ R \not\subseteq S \text{ \& } n \in S}} (T_C(S) - m_C(S)) \right\}.$$

1. $Q_C^R \geq 0$ for every $R \in 2_{\neq \emptyset}^{[n]}$ s.t. $\{n\} \subsetneq R$.

2. If $t_n > \mu^{\{n\}}$, then there exists $R \in 2_{\neq \emptyset}^{[n]}$ s.t. $\{n\} \subsetneq R$, $\mu^R > 0$ and $Q_C^R > 0$.

Let $R \in 2_{\neq \emptyset}^{[n]}$ s.t. $\{n\} \subsetneq R$ and let $\mu \in [0, \mu^R]$. For every $R' \in 2_{\neq \emptyset}^{[n]} \setminus \{R, \{n\}\}$, let $\mu^{R'} \triangleq \mu^{R'}$ and let $\mu'^R \triangleq \mu^R - \mu \geq 0$ and $\mu'^{\{n\}} \triangleq \mu^{\{n\}} + \mu$. Define $C' \triangleq ((\mu'^R)_{R \in 2_{\neq \emptyset}^{[n]}}, ([t_j, T_j])_{j \in [n]})$.

3. If $\mu \leq Q_C^R$, then C' is normal. Furthermore, in this case $Q_{C'}^R = Q_C^R - \mu$, and $Q_{C'}^{R'} \leq Q_C^{R'}$ for every $R' \in 2_{\neq \emptyset}^{[n]}$ s.t. $\{n\} \subsetneq R'$.

4. If C' is satisfiable, then C is satisfiable.

Remark 5. The condition of Lemma 9(3) is actually also necessary, i.e. C' is normal iff $\mu \leq Q_C^R$.

Proof of Lemma 9. Part 1 follows directly from the fact that C is normal, and so $M_C(S) - t_C(S) \geq 0$ and $T_C(S) - m_C(S) \geq 0$ for every $S \in 2_{\neq \emptyset}^{[n]}$.

To prove Part 2, let $S_1 \triangleq \bigcup \{S \in 2_{\neq \emptyset}^{[n-1]} \mid M_C(S) = t_C(S)\} \subseteq 2_{\neq \emptyset}^{[n-1]}$ and $S_2 \triangleq [n] \cap \bigcap \{S \in 2_{\neq \emptyset}^{[n]} \mid n \in S \text{ \& } T_C(S) = m_C(S)\} \supseteq \{n\}$ (the intersection with $[n]$ has any effect only if $\{n\}$ is the sole element in the intersection defining S_2). We first show that there exists $R \subseteq S_2 \setminus S_1$ s.t. $\{n\} \subsetneq R$ and $\mu^R > 0$.

For ease of notation, we extend the definition of $m_C(S)$, $M_C(S)$, $t_C(S)$ and $T_C(S)$ also to the case $S = \emptyset$, via the same definition; we note that these all equal zero when $S = \emptyset$, as they are all defined by empty sums in this case. We note that if $S_1 \neq \emptyset$, then $M_C(S_1) = t_C(S_1)$ by Lemma 8(1), and if $S_1 = \emptyset$, then $M_C(S_1) = 0 = t_C(S_1)$ by definition.

We first consider the case in which $S_2 \neq [n]$. In this case, by Lemma 8(2), $T_C(S_2) = m_C(S_2)$. Let $S \triangleq S_1 \cap S_2 \subseteq [n-1]$. We note that $t_C(S_1) - t_C(S) = t_C(S_1 \setminus S) \leq M_C(S_1 \setminus S) = M_C(S_1) - \sum_{R \in 2_{\neq \emptyset}^{[n]} \setminus 2_{\neq \emptyset}^{[n]} \setminus S: R \cap (S_1 \setminus S) = \emptyset} \mu^R \leq M_C(S_1) - \sum_{R \in 2_{\neq \emptyset}^{S_2} \setminus 2_{\neq \emptyset}^{S_2} \setminus S} \mu^R = t_C(S_1) - \sum_{R \in 2_{\neq \emptyset}^{S_2} \setminus 2_{\neq \emptyset}^{S_2} \setminus S} \mu^R$; therefore, $\sum_{R \in 2_{\neq \emptyset}^{S_2} \setminus 2_{\neq \emptyset}^{S_2} \setminus S} \mu^R \leq t_C(S) \leq T_C(S)$. Hence, and as $T_n \geq t_n > \mu^{\{n\}}$, we have that $m_C(S_2) - \sum_{R \in 2_{\neq \emptyset}^{S_2} \setminus 2_{\neq \emptyset}^{S_2} \setminus S} \mu^R - \mu^{\{n\}} - \sum_{R \in 2_{\neq \emptyset}^{S_2} \setminus S: \{n\} \subsetneq R} \mu^R = m_C(S_2 \setminus (S \cup \{n\})) \leq T_C(S_2 \setminus (S \cup \{n\})) = T_C(S_2) - T_C(S) - T_C(\{n\}) = m_C(S_2) - T_C(S) - T_n < m_C(S_2) - \sum_{R \in 2_{\neq \emptyset}^{S_2} \setminus 2_{\neq \emptyset}^{S_2} \setminus S} \mu^R - \mu^{\{n\}}$. Therefore, $\sum_{R \in 2_{\neq \emptyset}^{S_2} \setminus S: \{n\} \subsetneq R} \mu^R > 0$, and so there exists $R \subseteq S_2 \setminus S = S_2 \setminus S_1$ s.t. $\{n\} \subsetneq R$ and $\mu^R > 0$, as required.

We now consider the case in which $S_2 = [n]$. Note that $M_C(S_1 \cup \{n\}) \geq t_C(S_1 \cup \{n\}) = t_C(S_1) + t_n = M_C(S_1) + t_n > M_C(S_1) + \mu^{\{n\}} = M_C(S_1 \cup \{n\}) - \sum_{R \in 2_{\neq \emptyset}^{[n] \setminus S_1} : \{n\} \subsetneq R} \mu^R$; therefore, $\sum_{R \in 2_{\neq \emptyset}^{[n] \setminus S_1} : \{n\} \subsetneq R} \mu^R > 0$, and so there exists $R \subseteq [n] \setminus S_1 = S_2 \setminus S_1$ s.t. $\{n\} \subsetneq R$ and $\mu^R > 0$, as required.

Either way, there exists $R \subseteq S_2 \setminus S_1$ s.t. $\{n\} \subsetneq R$ and $\mu^R > 0$. Therefore, for every $S \in 2_{\neq \emptyset}^{[n-1]}$ s.t. $S \cap R \neq \emptyset$, we have $S \not\subseteq S_1$ and so $M_C(S) \neq t_C(S)$ and by normality of C , $M_C(S) > t_C(S)$; for every $S \in 2_{\neq \emptyset}^{[n]}$ s.t. $n \in S$ and $R \not\subseteq S$, we have $S_2 \not\subseteq S$ and so $T_C(S) \neq m_C(S)$ and by normality of C , $T_C(S) > m_C(S)$. By both of these, $Q_C^R > 0$ and the proof of Part 2 is complete.

We move on to Part 3; let $S \in 2_{\neq \emptyset}^{[n]}$. If $R \subseteq S$ (and so also $n \in S$) or both $R \not\subseteq S$ and $n \notin S$, then by normality of C ,

$$T_{C'}(S) = T_C(S) \geq m_C(S) = \sum_{R \in 2_{\neq \emptyset}^S} \mu^R = \sum_{R \in 2_{\neq \emptyset}^S} \mu'^R = m_{C'}(S);$$

otherwise, $R \not\subseteq S$ and $n \in S$, and by definition of μ and of Q_C^R ,

$$T_{C'}(S) = T_C(S) \geq m_C(S) + Q_C^R \geq m_C(S) + \mu = \sum_{R \in 2_{\neq \emptyset}^S} \mu'^R = m_{C'}(S).$$

If $S \cap R = \emptyset$ (and so also $n \notin S$) or both $S \cap R \neq \emptyset$ and $n \in S$, then by normality of C ,

$$t_{C'}(S) = t_C(S) \leq M_C(S) = \sum_{R \in 2_{\neq \emptyset}^{[n] \setminus S}} \mu^R = \sum_{R \in 2_{\neq \emptyset}^{[n] \setminus S}} \mu'^R = M_{C'}(S);$$

otherwise, $S \cap R \neq \emptyset$ and $n \notin S$, and by definition of μ and of Q_C^R ,

$$t_{C'}(S) = t_C(S) \leq M_C(S) - Q_C^R \leq M_C(S) - \mu = \sum_{R \in 2_{\neq \emptyset}^{[n] \setminus S}} \mu'^R = M_{C'}(S).$$

Therefore, C' is normal.

For every $S \in 2_{\neq \emptyset}^{[n-1]}$ s.t. $S \cap R \neq \emptyset$, we have that $M_{C'}(S) = M_C(S) - \mu$; for every $S \in 2_{\neq \emptyset}^{[n]}$ s.t. $R \not\subseteq S$ and $n \in S$, we have that $m_{C'}(S) = m_C(S) + \mu$. Therefore,

$$\begin{aligned} Q_{C'}^R &= \min \left\{ \min_{\substack{S \in 2_{\neq \emptyset}^{[n-1]} : \\ S \cap R \neq \emptyset}} (M_{C'}(S) - t_{C'}(S)), \min_{\substack{S \in 2_{\neq \emptyset}^{[n]} : \\ R \not\subseteq S \text{ \& } n \in S}} (T_{C'}(S) - m_{C'}(S)) \right\} = \\ &= \min \left\{ \min_{\substack{S \in 2_{\neq \emptyset}^{[n-1]} : \\ S \cap R \neq \emptyset}} (M_C(S) - \mu - t_C(S)), \min_{\substack{S \in 2_{\neq \emptyset}^{[n]} : \\ R \not\subseteq S \text{ \& } n \in S}} (T_C(S) - m_C(S) - \mu) \right\} = \\ &= Q_C^R - \mu. \end{aligned}$$

For every $S \in 2_{\neq \emptyset}^{[n-1]}$, we have $M_{C'}(S) \in \{M_C(S), M_C(S) - \mu\}$ (as shown above, depending on whether or not both $S \cap R \neq \emptyset$ and $n \notin S$); for every $S \in 2_{\neq \emptyset}^{[n]}$ s.t. $n \in S$, we have that $m_{C'}(S) \in \{m_C(S), m_C(S) + \mu\}$ (as shown above, depending on whether or not both $R \not\subseteq S$ and

$n \in S$). Therefore, for every $R' \in 2_{\neq \emptyset}^{[n]}$ s.t. $\{n\} \subsetneq R'$,

$$\begin{aligned} Q_{C'}^{R'} &= \min \left\{ \min_{\substack{S \in 2_{\neq \emptyset}^{[n-1]}: \\ S \cap R' \neq \emptyset}} (M_{C'}(S) - t_{C'}(S)), \min_{\substack{S \in 2_{\neq \emptyset}^{[n]}: \\ R' \not\subseteq S \text{ \& } n \in S}} (T_{C'}(S) - m_{C'}(S)) \right\} \leq \\ &\leq \min \left\{ \min_{\substack{S \in 2_{\neq \emptyset}^{[n-1]}: \\ S \cap R' \neq \emptyset}} (M_C(S) - t_C(S)), \min_{\substack{S \in 2_{\neq \emptyset}^{[n]}: \\ R' \not\subseteq S \text{ \& } n \in S}} (T_C(S) - m_C(S)) \right\} = \\ &= Q_C^{R'}. \end{aligned}$$

Therefore, the proof of Part 3 is complete.

We conclude by proving Part 4. As C' is satisfiable, by definition there exist $(\mu_j^{R'})_{j \in [n]}^{R' \in 2_{\neq \emptyset}^{[n]}}$ s.t. $(\mu_j^{R'})_{j \in [n]} \in \mu^{R'} \cdot \Delta^{R'}$ for every $R' \in 2_{\neq \emptyset}^{[n]}$ and $\sum_{R' \in 2_{\neq \emptyset}^{[n]}} \mu_j^{R'} \in [t_j, T_j]$ for every $j \in [n]$. For every $(j, R') \in [n] \times 2_{\neq \emptyset}^{[n]}$, if $j \neq n$ or $R' \notin \{R, \{n\}\}$, let $\mu_j^{R'} \triangleq \mu_j^{R'}$; let $\mu_n^R \triangleq \mu_n^R + \mu$ and $\mu_n^{\{n\}} \triangleq \mu_n^{\{n\}} - \mu$. (As $\mu_n^{\{n\}} = \mu^{\{n\}} = \mu^{\{n\}} + \mu$, we have that $\mu_n^{\{n\}} \in \mathbb{R}_{\geq}$.)

For every $R' \in 2_{\neq \emptyset}^{[n]} \setminus \{R, \{n\}\}$, by definition $(\mu_j^{R'})_{j \in [n]} = (\mu_j^{R'})_{j \in [n]} \in \mu^{R'} \cdot \Delta^{R'} = \mu^R \cdot \Delta^{R'}$. Furthermore, as $(\mu_j^{R'})_{j \in [n]} \in \mu^{R'} \cdot \Delta^{R'}$ and by definition of $(\mu_j^R)_{j \in [n]}$ and as $n \in R$, we have that $(\mu_j^R)_{j \in [n]} \in (\mu^R + \mu) \cdot \Delta^R = \mu^R \cdot \Delta^R$. Similarly, as $(\mu_j^{\{n\}})_{j \in [n]} \in \mu^{\{n\}} \cdot \Delta^{\{n\}}$ and by definition of $(\mu_j^{\{n\}})_{j \in [n]}$, we have that $(\mu_j^{\{n\}})_{j \in [n]} \in (\mu^{\{n\}} - \mu) \cdot \Delta^{\{n\}} = \mu^{\{n\}} \cdot \Delta^{\{n\}}$. Finally, $\sum_{R' \in 2_{\neq \emptyset}^{[n]}} \mu_j^{R'} = \sum_{R' \in 2_{\neq \emptyset}^{[n]}} \mu_j^{R'} \in [t_j, T_j]$ for every $j \in [n]$, and the proof is complete. \square

Lemma 10 (Distributing All Mass but $\mu^{\{n\}}$ among $[n-1]$). *Let $C = ((\mu^R)_{R \in 2_{\neq \emptyset}^{[n]}}, ([t_j, T_j])_{j \in [n]})$ be a normal distribution constraint s.t. $\mu^{\{n\}} \geq t_n$. We say that condition D_C holds if $T_C(S) \geq m_C(S \cup \{n\}) - \mu^{\{n\}}$ for every $S \in 2_{\neq \emptyset}^{[n-1]}$.*

1. If $T^n = \mu^{\{n\}}$, then condition D_C holds.

2. If $Q_C^R = 0$ for every $R \in 2_{\neq \emptyset}^{[n]}$ s.t. $\{n\} \subsetneq R$ and $\mu^R > 0$, then condition D_C holds.

For every $R \in 2_{\neq \emptyset}^{[n-1]}$, let $\mu^{R'} \triangleq \mu^R + \mu^{R \cup \{n\}}$. Define $C' \triangleq ((\mu^{R'})_{R \in 2_{\neq \emptyset}^{[n-1]}}, ([t_j, T_j])_{j \in [n-1]})$.

3. If condition D_C holds, then C' is normal.

4. If C' is satisfiable, the C is satisfiable.

Remark 6. Once again, the condition of Lemma 10(3) is actually also necessary, i.e. C' is normal iff condition D_C holds.

Proof of Lemma 10. Part 1 holds as for every $S \in 2_{\neq \emptyset}^{[n-1]}$, $T_C(S) = T_C(S \cup \{n\}) - T^n \geq m_C(S \cup \{n\}) - T^n = m_C(S \cup \{n\}) - \mu^{\{n\}}$.

To prove Part 2, define S_1 and S_2 as in the proof of Lemma 9(2); as in that proof, it suffices to show that if condition D_C does not hold, then there exists $R \subseteq S_2 \setminus S_1$ s.t. $\{n\} \subsetneq R$ and $\mu^R > 0$. As in that proof, we extend the definition of $m_C(S)$, $M_C(S)$, $t_C(S)$ and $T_C(S)$ also to the case $S = \emptyset$. By Part 1, $T_n > \mu^{\{n\}}$. If $S_2 \neq [n]$, then the proof follows as in the proof of Lemma 9(2) (as that proof only uses the fact that $\mu^{\{n\}} < T_n$ when $S_2 \neq [n]$, and does not rely on the inequality $\mu^{\{n\}} < t_n$ for this case). It therefore remains to consider the case in which $S_2 = [n]$.

Recall that if $S_1 \neq \emptyset$, then $M_C(S_1) = t_C(S_1)$ by Lemma 8(1), and if $S_1 = \emptyset$, then $M_C(S_1) = 0 = t_C(S_1)$ by definition. As condition D_C does not hold, there exists $S \in 2_{\neq \emptyset}^{[n-1]}$ s.t. $T_C(S) < m_C(S \cup \{n\}) - \mu^{\{n\}}$.

Let $S' \triangleq S_1 \cap S \subseteq [n-1]$. We note that $t_C(S_1) - t_C(S') = t_C(S_1 \setminus S') \leq M_C(S_1 \setminus S') = M_C(S_1) - \sum_{R \in 2_{\neq \emptyset}^{[n]} \setminus 2_{\neq \emptyset}^{[n] \setminus S'} : R \cap (S_1 \setminus S') = \emptyset} \mu^R \leq M_C(S_1) - \sum_{R \in 2_{\neq \emptyset}^{S \cup \{n\}} \setminus 2_{\neq \emptyset}^{S \cup \{n\} \setminus S'}} \mu^R = t_C(S_1) - \sum_{R \in 2_{\neq \emptyset}^{S \cup \{n\}} \setminus 2_{\neq \emptyset}^{S \cup \{n\} \setminus S'}} \mu^R$; therefore, $\sum_{R \in 2_{\neq \emptyset}^{S \cup \{n\}} \setminus 2_{\neq \emptyset}^{S \cup \{n\} \setminus S'}} \mu^R \leq t_C(S') \leq T_C(S')$. Hence, by definition of S we have that $m_C(S \cup \{n\}) - \sum_{R \in 2_{\neq \emptyset}^{S \cup \{n\}} \setminus 2_{\neq \emptyset}^{(S \cup \{n\}) \setminus S'}} \mu^R - \mu^{\{n\}} - \sum_{R \in 2_{\neq \emptyset}^{(S \cup \{n\}) \setminus S'} : \{n\} \subsetneq R} \mu^R = m_C(S \setminus S') \leq T_C(S \setminus S') = T_C(S) - T_C(S') < m_C(S \cup \{n\}) - T_C(S') - \mu^{\{n\}} \leq m_C(S \cup \{n\}) - \sum_{R \in 2_{\neq \emptyset}^{S \cup \{n\}} \setminus 2_{\neq \emptyset}^{(S \cup \{n\}) \setminus S'}} \mu^R - \mu^{\{n\}}$. Therefore, $\sum_{R \in 2_{\neq \emptyset}^{(S \cup \{n\}) \setminus S'} : \{n\} \subsetneq R} \mu^R > 0$, and so there exists $R \subseteq (S \cup \{n\}) \setminus S' \subseteq S_2 \setminus S_1$ s.t. $\{n\} \subsetneq R$ and $\mu^R > 0$, as required, and the proof of Part 2 is complete.

We move on to Part 3. For every $S \in 2_{\neq \emptyset}^{[n-1]}$, as condition D_C holds,

$$T_{C'}(S) = T_C(S) \geq m_C(S \cup \{n\}) - \mu^{\{n\}} = \sum_{R \in 2_{\neq \emptyset}^S} (\mu^R + \mu^{R \cup \{n\}}) = m_{C'}(S);$$

furthermore,

$$t_{C'}(S) = t_C(S) \leq M_C(S) = \sum_{R \in 2_{\neq \emptyset}^{[n]} \setminus 2_{\neq \emptyset}^{[n] \setminus S}} \mu^R = \sum_{R \in 2_{\neq \emptyset}^{[n-1]} \setminus 2_{\neq \emptyset}^{[n-1] \setminus S}} (\mu^R + \mu^{R \cup \{n\}}) = M_{C'}(S).$$

Therefore, C' is normal, as required.

We conclude by proving Part 4. As C' is satisfiable, by definition there exist $(\mu_j^{R'})_{j \in [n-1]}^{R' \in 2_{\neq \emptyset}^{[n-1]}}$ s.t. $(\mu_j^{R'})_{j \in [n-1]} \in \mu'^R \cdot \Delta^R$ for every $R \in 2_{\neq \emptyset}^{[n]}$ and $\sum_{R \in 2_{\neq \emptyset}^{[n-1]}} \mu_j^{R'} \in [t_j, T_j]$ for every $j \in [n-1]$. For every $R \in 2_{\neq \emptyset}^{[n-1]}$, if $\mu'^R = 0$, then we define $\mu_j^R \triangleq 0$ and $\mu_j^{R \cup \{n\}} \triangleq 0$ for every $j \in [n-1]$; otherwise, we define $\mu_j^R \triangleq \frac{\mu^R}{\mu'^R} \cdot \mu_j^{R'}$ and $\mu_j^{R \cup \{n\}} \triangleq \frac{\mu^{R \cup \{n\}}}{\mu'^R} \cdot \mu_j^{R'}$ for every $j \in [n-1]$. We further define $\mu_n^R \triangleq 0$ for every $R \in 2_{\neq \emptyset}^{[n]} \setminus \{n\}$, $\mu_n^{\{n\}} \triangleq \mu^{\{n\}}$ and $\mu_j^{\{n\}} \triangleq 0$ for every $j \in [n-1]$.

For every $R \in 2_{\neq \emptyset}^{[n-1]}$, if $\mu'^R = 0$ then by definition $(\mu_j^R)_{j \in [n]} \equiv 0 \in 0 \cdot \Delta^R = \mu^R \cdot \Delta^R$ and similarly $(\mu_j^{R \cup \{n\}})_{j \in [n]} \equiv 0 \in 0 \cdot \Delta^R = \mu^{R \cup \{n\}} \cdot \Delta^{R \cup \{n\}}$; otherwise, as $(\mu_j^{R'})_{j \in [n-1]} \in \mu'^R \cdot \Delta^R$ and by definition of $(\mu_j^R)_{j \in [n]}$ and $(\mu_j^{R \cup \{n\}})_{j \in [n]}$, we have that $(\mu_j^R)_{j \in [n]} \in \frac{\mu^R}{\mu'^R} \mu'^R \cdot \Delta^R = \mu^R \cdot \Delta^R$ and similarly $(\mu_j^{R \cup \{n\}})_{j \in [n]} \in \frac{\mu^{R \cup \{n\}}}{\mu'^R} \mu'^R \cdot \Delta^R = \mu^{R \cup \{n\}} \cdot \Delta^R \subseteq \mu^{R \cup \{n\}} \cdot \Delta^{R \cup \{n\}}$. Furthermore, by definition $(\mu_j^{\{n\}})_{j \in [n]} \in \mu^{\{n\}} \cdot \Delta^{\{n\}}$. Finally, $\sum_{R \in 2_{\neq \emptyset}^{[n]}} \mu_j^R = \sum_{R \in 2_{\neq \emptyset}^{[n-1]}} \mu_j^{R'} \in [t_j, T_j]$ for every $j \in [n-1]$, and $\sum_{R \in 2_{\neq \emptyset}^{[n]}} \mu_n^R = \mu_n^{\{n\}} = \mu^{\{n\}} \in [t_n, T_n]$ (where $\mu^{\{n\}} = m_C(\{n\}) \leq T_C(\{n\}) = T_n$ by normality of C) and the proof is complete. \square

Proof of Lemma 7. Let $C = ((\mu^R)_{R \in 2_{\neq \emptyset}^{[n]}}, ([t_j, T_j])_{j \in [n]})$ be a normal distribution constraint. We prove the claim by induction on $n \in \mathbb{N}$.

(Outer induction) Base: For $n = 1$, we have by definition that $m_C(\{1\}) = \mu^{\{1\}} = M_C(\{1\})$, and so $t_1 = t_C(\{1\}) \leq M_C(\{1\}) = \mu^{\{1\}} = m_C(\{1\}) \leq T_C(\{1\}) = T_1$. Therefore, setting $\mu_1^{\{1\}} \triangleq \mu^{\{1\}}$ completes the proof of the (outer) induction base.

(Outer induction) Step: Let $n > 1$ and assume that the lemma holds for $n-1$. We prove the induction step by full induction on $|\{R \in 2_{\neq \emptyset}^{[n]} \mid \{n\} \subsetneq 2_{\neq \emptyset}^{[n]} \text{ \& } \mu^R > 0 \text{ \& } Q_C^R > 0\}|$.

(Inner induction) Base: If $|\{R \in 2_{\neq \emptyset}^{[n]} \mid \{n\} \subsetneq R \text{ \& } \mu^R > 0 \text{ \& } Q_C^R > 0\}| = 0$, then by Lemma 9(2), $t_n \leq \mu^{\{n\}}$, and by Lemma 10(2), condition D_C holds. Therefore, by Lemma 10(3), C' as defined in Lemma 10 is normal, and by the (outer) induction hypothesis for $n - 1$, C' is satisfiable. By Lemma 10(4), C is satisfiable as well.

(Inner induction) Step: Assume that $|\{R \in 2_{\neq \emptyset}^{[n]} \mid \{n\} \subsetneq R \text{ \& } \mu^R > 0 \text{ \& } Q_C^R > 0\}| > 0$ and that the claim holds whenever this set is of smaller cardinality. Therefore, there exists $R \in 2_{\neq \emptyset}^{[n]}$ s.t. $\{n\} \subsetneq R$, $\mu^R > 0$ and $Q_C^R > 0$; let $\mu \triangleq \min\{Q_C^R, \mu^R\} > 0$, and define C' w.r.t. R and μ as in Lemma 9. By Lemma 9(3), C' is normal. If $\mu = \mu^R$, then $\mu^{R'} = 0$; otherwise, $\mu = Q_C^R$ and by Lemma 9(3), $Q_{C'}^R = Q_C^R - \mu = 0$. Either way, and by definition of C' and as by Lemma 9(3) $Q_{C'}^{R'} = 0$ whenever $Q_C^{R'} = 0$, we have that $|\{R' \in 2_{\neq \emptyset}^{[n]} \mid \{n\} \subsetneq R' \text{ \& } \mu^{R'} > 0 \text{ \& } Q_{C'}^{R'} > 0\}| \leq |\{R' \in 2_{\neq \emptyset}^{[n]} \mid \{n\} \subsetneq R' \text{ \& } \mu^{R'} > 0 \text{ \& } Q_C^{R'} > 0\} \setminus \{R\}| = |\{R' \in 2_{\neq \emptyset}^{[n]} \mid \{n\} \subsetneq R' \text{ \& } \mu^{R'} > 0 \text{ \& } Q_C^{R'} > 0\}| - 1$ and so, by the (inner) induction hypothesis, C' is satisfiable. By Lemma 9(4), C is satisfiable as well and the proof is complete. \square

A.1.5 Existence

Lemma 11 ($h_G = \text{Max}_{S \in 2_{\neq \emptyset}^{[n]}} E_G(S)$). *Let $G = ((f_j)_{j=1}^n; (\mu^R)_{R \in 2_{\neq \emptyset}^{[n]}})$ be a resource selection game.*

1. $\text{Max}_{S \in 2_{\neq \emptyset}^{[n]}} E_G(S) \in \mathbb{R}$ is well defined.
2. If f_1, \dots, f_n are continuous, then $h_G = \text{Max}_{S \in 2_{\neq \emptyset}^{[n]}} E_G(S)$.

In both parts, the value undefined is treated as $-\infty$ for comparisons by the Max operator.

Proof. To show Part 1, note that by Corollary 2(1), $E_G(\{1\}) = \text{Equalize}_{f_1}(\mu^{\{1\}}) = f_1(\mu^{\{1\}}) \in \mathbb{R}$; therefore, $E_G(\{1\}) \in \mathbb{R}$, and so $\text{Max}_{S \in 2_{\neq \emptyset}^{[n]}} E_G(S) \in \mathbb{R}$, as required.

Define $A \triangleq \arg \text{Max}_{S \in 2_{\neq \emptyset}^{[n]}} E_G(S)$. Before proving Part 2, we first show that for every $S \in A$ and $M' \in M_G(S)$, if $(f_j)_{j \in S \setminus M'}$ are continuous, then also $S \setminus M' \in A$. Let, therefore, $S \in A$ and let $M' \in M_G(S)$ s.t. $(f_j)_{j \in S \setminus M'}$ are continuous. By definition of M' , we have both that $M' \subsetneq S$ (see Lemma 3) and that $\text{Equalize}_{f_k: k \in M'}(\mu) \neq E_G(S)$ for every $\mu \leq \sum_{R \in 2_{\neq \emptyset}^S \setminus 2_{\neq \emptyset}^{S \setminus M'}} \mu^R$. By Part 1, $E_G(S) \in \mathbb{R}$, and so by definition there exist $(\mu_j)_{j \in S} \in \mathbb{R}_{\geq}^S$ s.t. $\sum_{j \in S} \mu_j = \sum_{R \in 2_{\neq \emptyset}^S} \mu^R$ and $f_j(\mu_j) = E_G(S)$ for every $j \in S$. Therefore, $\text{Equalize}_{f_k: k \in S \setminus M'}(\sum_{j \in S \setminus M'} \mu_j) = E_G(S)$, and also $\text{Equalize}_{f_k: k \in M'}(\mu) = E_G(S)$, where $\mu \triangleq \sum_{j \in M'} \mu_j$.

As $M' \in M_G(S)$, we thus have that $\mu > \sum_{R \in 2_{\neq \emptyset}^S \setminus 2_{\neq \emptyset}^{S \setminus M'}} \mu^R$. Therefore, $\sum_{R \in 2_{\neq \emptyset}^{S \setminus M'}} \mu^R = \sum_{R \in 2_{\neq \emptyset}^S} \mu^R - \sum_{R \in 2_{\neq \emptyset}^S \setminus 2_{\neq \emptyset}^{S \setminus M'}} \mu^R > \sum_{R \in 2_{\neq \emptyset}^S} \mu^R - \mu = \sum_{j \in S} \mu_j - \mu = \sum_{j \in S} \mu_j - \sum_{j \in M'} \mu_j = \sum_{j \in S \setminus M'} \mu_j$. Recall that $\text{Equalize}_{f_k: k \in S \setminus M'}(\sum_{j \in S \setminus M'} \mu_j) = E_G(S) \in \mathbb{R}$; therefore, by continuity of $(f_k)_{k \in S \setminus M'}$ and by Lemma 2(2), we obtain that also $\text{Equalize}_{f_k: k \in S \setminus M'}(\sum_{R \in 2_{\neq \emptyset}^{S \setminus M'}} \mu^R) \in \mathbb{R}$. Hence, by Lemma 1 we conclude that $E_G(S \setminus M') = \text{Equalize}_{f_k: k \in S \setminus M'}(\sum_{R \in 2_{\neq \emptyset}^{S \setminus M'}} \mu^R) \geq \text{Equalize}_{f_k: k \in S \setminus M'}(\sum_{j \in S \setminus M'} \mu_j) = E_G(S)$, and so indeed $S \setminus M' \in A$, as required.

We conclude by proving Part 2. By definition $h_G \leq \text{Max}_{S \in 2_{\neq \emptyset}^{[n]}} E_G(S)$; we therefore have to show that $h_G \geq \text{Max}_{S \in 2_{\neq \emptyset}^{[n]}} E_G(S)$. Let $S \in A$. We iteratively define a series $(S_i)_{i=0}^k$, for $k \in \mathbb{N}$ to be determined, as follows:

- $S_0 \triangleq S$.

- If $M_G(S_i) = \emptyset$, then we set $k \triangleq i$ and conclude. Otherwise, choose $M_i \in M_G(S_i)$ arbitrarily, and set $S_{i+1} \triangleq S_i \setminus M_i$.

We now show by induction that $S_i \in A$ and $|S_i| \leq |S| - i$ for every i for which S_i is defined.

- Base: By definition, $S_0 = S \in A$ and $|S_0| \leq |S| - 0 = |S|$, as required.
- Step: Let $i > 0$ for which S_i is defined. By the induction hypothesis, $S_{i-1} \in A$; therefore, as shown above and by continuity of $(f_j)_{j \in S_{i-1} \setminus M_{i-1}}$, we have that $S_i = S_{i-1} \setminus M_{i-1} \in A$. Furthermore, as by definition $M_{i-1} \neq \emptyset$, we have by the induction hypothesis that $|S_i| = |S_{i-1}| - |M_{i-1}| \leq |S_{i-1}| - 1 \leq |S| - (i-1) - 1 = |S| - i$, as required.

We conclude that the process constructing $(S_i)_i$ indeed stops (i.e. k is well defined), and with $k < |S|$. By definition, $M_G(S_k) = \emptyset$, and as $S_k \in A$, by Part 1 we have $E_G(S_k) = \max_{S' \in 2_{\neq \emptyset}^{[n]}} E_G(S') \in \mathbb{R}$; therefore, $S_k \in D_G$. Therefore, $h_G \geq E_G(S_k) = \max_{S' \in 2_{\neq \emptyset}^{[n]}} E_G(S')$, as required. \square

We note that it can be shown that, in the context of Lemma 11(2), for every $S \in A$ s.t. $M_G(S) \neq \emptyset$, in fact $\bigcup M_G(S) \in M_G(S)$ and $M_G(S \setminus \bigcup M_G(S)) = \emptyset$. While this may be used to avoid the inductive construction concluding the proof of this lemma, the need to prove these facts would result in a considerably longer total length for the proof.

Lemma 12. *Let $G = ((f_j)_{j=1}^n; (\mu^R)_{R \in 2_{\neq \emptyset}^{[n]}})$ be a resource selection game s.t. f_1, \dots, f_n are continuous. For every $S \in \arg \max_{S' \in D_G} E_G(S')$, there exists a strategy profile s in the $|S|$ -resource selection game $G' \triangleq ((f_j)_{j \in S}; (\mu^R)_{R \in 2_{\neq \emptyset}^S})$, s.t. $h_j^s = h_G$ for every $j \in S$.*

Proof. For every $j \in S$, let $t_j \triangleq \min f_j^{-1}(h_G)$ and $T_j \triangleq \max f_j^{-1}(h_G)$ if $\sup f_j^{-1}(h_G) \neq \infty$ and $T_j \triangleq \sum_{R \in 2_{\neq \emptyset}^{[n]}} \mu^R$ otherwise (t_j and T_j are well defined by continuity of f_j and since $E_G(S) = h_G$); regardless of how we define T_j , we have that both $f_j(T_j) = h_G$ and $T_j \geq t_j$ (when $\sup f_j^{-1}(h_G) = \infty$, this is since $E_G(S) = h_G$ and since f_j is nondecreasing).

We now show that $C \triangleq ((\mu^R)_{R \in 2_{\neq \emptyset}^S}, ([t_j, T_j])_{j \in S})$ is a normal distribution constraint. (See Appendix A.1.4; we slightly abuse notation by treating S in the context of C as $[|S|]$, using an arbitrary isomorphism.) Let $S' \in 2_{\neq \emptyset}^S$. As $M(S) = \emptyset$, there exists $\mu \leq \sum_{R \in 2_{\neq \emptyset}^S \setminus 2_{\neq \emptyset}^{S \setminus S'}} \mu^R$ s.t. $\text{Equalize}_{f_k: k \in S'}(\mu) = E_G(S) = h_G$; therefore, $t_C(S') = \sum_{j \in S'} t_j \leq \mu \leq \sum_{R \in 2_{\neq \emptyset}^S \setminus 2_{\neq \emptyset}^{S \setminus S'}} \mu^R = M_C(S')$. Assume for contradiction that $\sum_{R \in 2_{\neq \emptyset}^{S'}} \mu^R > \sum_{j \in S'} T_j$. As $f_j(T_j) = h_G$ for every $j \in S'$, we have $\text{Equalize}_{f_k: k \in S'}(\sum_{j \in S'} T_j) = h_G \in \mathbb{R}$. By continuity of $(f_j)_{j \in S'}$ and by Lemma 2(2), we thus have that $E_G(S') \in \mathbb{R}$; therefore, there exist $(\mu_j)_{j \in S'}$ s.t. $\sum_{j \in S'} \mu_j = \sum_{R \in 2_{\neq \emptyset}^{S'}} \mu^R$ and $f_j(\mu_j) = E_G(S')$ for every $j \in S'$. As $\sum_{j \in S'} \mu_j = \sum_{R \in 2_{\neq \emptyset}^{S'}} \mu^R > \sum_{j \in S'} T_j$, there exists $j \in S'$ s.t. $\mu_j > T_j$; As $T_j < \mu_j \leq \sum_{R \in 2_{\neq \emptyset}^{S'}} \mu^R \leq \sum_{R \in 2_{\neq \emptyset}^{[n]}} \mu^R$, we have that $T_j = \max f_j^{-1}(h_G)$, and so $E_G(S') = f_j(\mu_j) > h_G = \max_{S'' \in 2_{\neq \emptyset}^{[n]}} E_G(S'')$ (where the last equality is by Lemma 11(2)) — a contradiction. Therefore, $m_C(S') = \sum_{R \in 2_{\neq \emptyset}^{S'}} \mu^R \leq \sum_{j \in S'} T_j = T_C(S')$.

As C is normal, by Lemma 7 it is satisfiable, and so there exist $(s_j(R))_{j \in S}^{R \in 2_{\neq \emptyset}^S}$ s.t. $s(R) \in \mu^R \cdot \Delta^R$ for every $R \in 2_{\neq \emptyset}^S$ and $\sum_{R \in 2_{\neq \emptyset}^S} s_j(R) \in [t_j, T_j]$ for every $j \in S$. By the former, s is a strategy profile in G' , and by the latter, for every $j \in S$ we have that $\mu_j^s \in [t_j, T_j]$, and so by definition of t_j and T_j and since f_j is nondecreasing, $h_j^s = f_j(\mu_j^s) = h_G$ and the proof is complete. \square

Lemma 13 ($E_G(P_G) = h_G$). Let $G = ((f_j)_{j=1}^n; (\mu^R)_{R \in 2_{\neq \emptyset}^{[n]}})$ be a resource selection game. If f_1, \dots, f_n are continuous, then $P_G \in \arg \text{Max}_{S \in D_G} E_G(S)$.

Proof. Define $A \triangleq \arg \text{Max}_{S \in D_G} E_G(S)$. By Lemma 3, $A \neq \emptyset$. Therefore, by definition of P_G , it is enough to show that $S' \cup S'' \in A$ for every $S', S'' \in A$. Let, therefore, $S', S'' \in A$; it is enough to show that $S' \cup S'' \in D_G$ and that $E_G(S' \cup S'') = h_G$.

By Lemma 12, there exists a strategy profile s' in the game $((f_j)_{j \in S'}; (\mu^R)_{R \in 2_{\neq \emptyset}^{S'}})$ s.t. $h_j^{s'} = h_G$ for every $j \in S'$; similarly, there exists a strategy profile s'' in the game $((f_j)_{j \in S''}; (\mu^R)_{R \in 2_{\neq \emptyset}^{S''}})$ s.t. $h_j^{s''} = h_G$ for every $j \in S''$. For every $R \in 2_{\neq \emptyset}^{S'}$, we define $s(R) \triangleq s'(R)$ and set $\tilde{\mu}^R \triangleq \mu^R$; for every $R \in 2_{\neq \emptyset}^{S''} \setminus 2_{\neq \emptyset}^{S'}$, we define $s_j(R) \triangleq s_j''(R)$ for every $j \in S'' \setminus S'$ and $s_j(R) \triangleq 0$ for every $j \in S' \cap S''$ and set $\tilde{\mu}^R \triangleq \mu^R - \sum_{j \in S' \cap S''} s_j(R)$; finally, for every $R \in 2_{\neq \emptyset}^{S' \cup S''} \setminus (2_{\neq \emptyset}^{S'} \cup 2_{\neq \emptyset}^{S''})$, we define $s(R) \triangleq 0$ and set $\tilde{\mu}^R \triangleq 0$. By definition, s is a consumption profile in the game $((f_j)_{j \in S' \cup S''}; (\tilde{\mu}^R)_{R \in 2_{\neq \emptyset}^{S' \cup S''}})$. Note that for every $R \in 2_{\neq \emptyset}^{S' \cup S''}$, we have that $\tilde{\mu}^R \leq \mu^R$.

For every $j \in S'$, by definition of s we have $\mu_j^s = \mu_j^{s'}$ and so $f_j(\mu_j^s) = f_j(\mu_j^{s'}) = h_j^{s'} = h_G$. For every $j \in S'' \setminus S'$, by definition we have $\mu_j^s = \mu_j^{s''}$ and so $f_j(\mu_j^s) = f_j(\mu_j^{s''}) = h_j^{s''} = h_G$. Therefore, $h_G = \text{Equalize}_{f_k: k \in S' \cup S''}(\sum_{j \in S' \cup S''} \mu_j^s) = \text{Equalize}_{f_k: k \in S' \cup S''}(\sum_{R \in 2_{\neq \emptyset}^{S' \cup S''}} \tilde{\mu}^R)$. As $\sum_{R \in 2_{\neq \emptyset}^{S' \cup S''}} \tilde{\mu}^R \leq \sum_{R \in 2_{\neq \emptyset}^{S' \cup S''}} \mu^R$, we have by continuity of $(f_k)_{k \in S' \cup S''}$, by Lemma 3 and by Lemma 2(2) that $E_G(S' \cup S'') \in \mathbb{R}$. Therefore, by Lemma 1, we obtain $E_G(S' \cup S'') \geq \text{Equalize}_{f_k: k \in S' \cup S''}(\sum_{R \in 2_{\neq \emptyset}^{S' \cup S''}} \tilde{\mu}^R) = h_G$. Thus, by Lemma 11(2), $E_G(S' \cup S'') = h_G$. It therefore remains to show that $S' \cup S'' \in D_G$.

We have to show that for every $S \in 2_{\neq \emptyset}^{S' \cup S''}$, there exists $\mu \leq \sum_{R \in 2_{\neq \emptyset}^{S' \cup S''} \setminus 2_{\neq \emptyset}^{(S' \cup S'') \setminus S}} \mu^R$ s.t. $\text{Equalize}_{f_k: k \in S}(\mu) = E_G(S' \cup S'')$; let therefore $S \in 2_{\neq \emptyset}^{S' \cup S''}$. Define $\mu \triangleq \sum_{j \in S} \mu_j^s$. As $f_j(\mu_j^s) = h_G$ for every $j \in S' \cup S''$, we have $\text{Equalize}_{f_k: k \in S}(\mu) = h_G = E_G(S' \cup S'')$. By definition of s , we have that $\mu = \sum_{j \in S} \mu_j^s \leq \sum_{R \in 2_{\neq \emptyset}^{S' \cup S''} \setminus 2_{\neq \emptyset}^{(S' \cup S'') \setminus S}} \tilde{\mu}^R \leq \sum_{R \in 2_{\neq \emptyset}^{S' \cup S''} \setminus 2_{\neq \emptyset}^{(S' \cup S'') \setminus S}} \mu^R$, and the proof is complete. \square

Proof of Lemma 5. Part 1 follows directly from Lemmas 12 and 13. We therefore prove Part 2. Assume for contradiction that $h_{G'} \geq h_G$; recall that by definition $P_{G'} \subseteq [n] \setminus P_G$ and so $P_{G'}$ and P_G are disjoint. As by Lemma 3, $P_{G'} \neq \emptyset$, we aim to obtain a contradiction by showing that $P_{G'} \subseteq P_G$.

By Lemma 13, $P_G \in \arg \text{Max}_{S \in D_G} E_G(S)$; therefore, we have by definition that $h_G = E_G(P_G) = \text{Equalize}_{f_k: k \in P_G}(\sum_{R \in 2_{\neq \emptyset}^{P_G}} \mu^R)$. By Lemma 3, $h_G \in \mathbb{R}$ and so there exist $(\mu_j)_{j \in P_G} \in \mathbb{R}_{\geq}^{P_G}$ s.t. $\sum_{j \in P_G} \mu_j = \sum_{R \in 2_{\neq \emptyset}^{P_G}} \mu^R$ and $f_j(\mu_j) = h_G$ for every $j \in P_G$.

Similarly, by Lemma 13, $P_{G'} \in \arg \text{Max}_{S \in D_{G'}} E_{G'}(S)$; therefore, and by definition of G' , we have that $h_{G'} = E_{G'}(P_{G'}) = \text{Equalize}_{f_k: k \in P_{G'}}(\sum_{R \in 2_{\neq \emptyset}^{P_G \cup P_{G'}} \setminus 2_{\neq \emptyset}^{P_G}} \mu^R)$. By Lemma 3, $h_{G'} \in \mathbb{R}$ and so there exist $(\mu'_j)_{j \in P_{G'}} \in \mathbb{R}_{\geq}^{P_{G'}}$ s.t. $\sum_{j \in P_{G'}} \mu'_j = \sum_{R \in 2_{\neq \emptyset}^{P_G \cup P_{G'}} \setminus 2_{\neq \emptyset}^{P_G}} \mu^R$ and $f_j(\mu'_j) = h_{G'} \geq h_G$ for every $j \in P_{G'}$. Let $j \in P_{G'}$. As f_j is nondecreasing, by Corollary 2(1) and by definition of h_G , we have $f_j(0) \leq f_j(\mu^{\{j\}}) = E_G(\{j\}) \leq h_G$. By continuity of f_j and by the intermediate value theorem, there thus exists $\mu_j \in [0, \mu'_j]$ s.t. $f_j(\mu_j) = h_G$.

As $f_j(\mu_j) = h_G$ for every $j \in P_G \cup P_{G'}$, by definition $\text{Equalize}_{f_k: k \in P_G \cup P_{G'}}(\sum_{j \in P_G \cup P_{G'}} \mu_j) = h_G$. As $\sum_{j \in P_G \cup P_{G'}} \mu_j = \sum_{j \in P_G} \mu_j + \sum_{j \in P_{G'}} \mu_j \leq \sum_{j \in P_G} \mu_j + \sum_{j \in P_{G'}} \mu'_j = \sum_{R \in 2_{\neq \emptyset}^{P_G}} \mu^R + \sum_{R \in 2_{\neq \emptyset}^{P_G \cup P_{G'}} \setminus 2_{\neq \emptyset}^{P_G}} \mu^R = \sum_{R \in 2_{\neq \emptyset}^{P_G \cup P_{G'}}} \mu^R$, we have by Lemma 2(2) that $E_G(P \cup P_{G'}) \in \mathbb{R}$ and

therefore, by Lemma 1, $E_G(P_G \cup P_{G'}) \geq h_G$. Therefore, by definition of P_G , in order to show that $P_{G'} \subseteq P_G$ and complete the proof, it is enough to show that $M_G(P_G \cup P_{G'}) = \emptyset$.

Let $S \in 2_{\neq \emptyset}^{P_G \cup P_{G'}}$. By Lemma 13, $M_G(P_G) = \emptyset$ and so, if $S \cap P_G \neq \emptyset$, then there exists $\mu'' \leq \sum_{R \in 2_{\neq \emptyset}^{P_G} \setminus 2_{\neq \emptyset}^{P_G \setminus (S \cap P_G)}} \mu^R$ s.t. $\text{Equalize}_{f_k: k \in S \cap P_G}(\mu'') = E_G(P_G) = h_G$. If $S \cap P_G = \emptyset$, then set $\mu'' \triangleq 0$.

Similarly, by Lemma 13, $M_{G'}(P_{G'}) = \emptyset$ and so, if $S \cap P_{G'} \neq \emptyset$, then there exists $\mu' \leq \sum_{R \in 2_{\neq \emptyset}^{P_G \cup P_{G'}} \setminus 2_{\neq \emptyset}^{P_G \cup P_{G'} \setminus (S \cap P_{G'})}} \mu^R$ s.t. $\text{Equalize}_{f_k: k \in S \cap P_{G'}}(\mu') = E_{G'}(P_{G'}) = h_{G'}$. As $f_j(\mu_j) = h_G$ for every $j \in P_{G'}$, we also have in this case that $\text{Equalize}_{f_k: k \in S \cap P_{G'}}(\sum_{j \in S \cap P_{G'}} \mu_j) = h_G$. By both of these and by Lemma 1, $\text{Equalize}_{f_k: k \in S \cap P_{G'}}(\min\{\mu', \sum_{j \in S \cap P_{G'}} \mu_j\}) = \min\{h_{G'}, h_G\} = h_G$ in this case. If $S \cap P_{G'} = \emptyset$, then set $\mu' \triangleq 0$.

Define $\mu \triangleq \mu'' + \min\{\mu', \sum_{j \in S \cap P_{G'}} \mu_j\}$. By definition of μ (and by Corollary 2(2) if neither $S \cap P_G = \emptyset$ nor $S \cap P_{G'} = \emptyset$), we have that $\text{Equalize}_{f_k: k \in S}(\mu) = h_G$; it is therefore enough to show that $\mu \leq \sum_{R \in 2_{\neq \emptyset}^{P_G \cup P_{G'}} \setminus 2_{\neq \emptyset}^{P_G \cup P_{G'} \setminus S}} \mu^R$ in order to complete the proof. Indeed, since P_G and $P_{G'}$ are disjoint, we obtain that $\mu = \mu'' + \min\{\mu', \sum_{j \in S \cap P_{G'}} \mu_j\} \leq \mu'' + \mu' \leq \sum_{R \in 2_{\neq \emptyset}^{P_G} \setminus 2_{\neq \emptyset}^{P_G \setminus (S \cap P_G)}} \mu^R + \sum_{R \in 2_{\neq \emptyset}^{P_G \cup P_{G'}} \setminus 2_{\neq \emptyset}^{P_G \cup P_{G'} \setminus (S \cap P_{G'})}} \mu^R \leq \sum_{R \in 2_{\neq \emptyset}^{P_G \cup P_{G'}} \setminus 2_{\neq \emptyset}^{P_G \cup P_{G'} \setminus S}} \mu^R$, as required. \square

A.2 Proof of the Theorems and Corollary from Section 4

Proof of Theorem 1. We prove by full induction on n that in every n -resource selection game $G = ((f_j)_{j=1}^n; (\mu^R)_{R \in 2_{\neq \emptyset}^{[n]}})$ s.t. f_1, \dots, f_n are continuous, there exists a Nash equilibrium s s.t. $\text{Max}_{j \in [n]} h_j^s \leq h_G$; from this claim, the theorem *a fortiori* follows. Let $n \in \mathbb{N}$ and assume that this claim holds for all smaller natural values of n . Let $G = ((f_j)_{j=1}^n; (\mu^R)_{R \in 2_{\neq \emptyset}^{[n]}})$ be an n -resource selection game.

By Lemma 5(1), there exists a strategy profile s'' in the $|P_G|$ -resource selection game $G'' \triangleq ((f_j)_{j \in P_G}; (\mu^R)_{R \in 2_{\neq \emptyset}^{P_G}})$ s.t. $h_j^{s''} = h_G$ for every $j \in P_G$. By definition of Nash equilibrium, s'' is a Nash equilibrium in G'' . If $P_G = [n]$, then $s \triangleq s''$ is a Nash equilibrium as required, and the proof of the induction step is complete. Assume, therefore, that $P_G \subsetneq [n]$; hence, and since $P_G \neq \emptyset$ by Lemma 3, by the induction hypothesis there exists a Nash equilibrium s' in the $[n] \setminus P_G$ -resource selection game $G' \triangleq ((f_j)_{j \in [n] \setminus P_G}; (\sum_{R \in O(R')} \mu^R)_{R' \in 2_{\neq \emptyset}^{[n] \setminus P_G}})$ (where $O(R')$ is defined as in Lemma 5(2)), s.t. $\text{Max}_{j \in [n] \setminus P_G} h_j^{s'} \leq h_{G'}$.

We construct a strategy profile s in G as follows: $s(R) \triangleq s''(R)$ for every $R \in 2_{\neq \emptyset}^{P_G}$, and for every $R' \in 2_{\neq \emptyset}^{[n] \setminus P_G}$, we pick $(s(R))_{R \in O(R')}$ arbitrarily among the tuples satisfying $s(R) \in \mu^R \cdot \Delta^{R'}$ for every $R \in O(R')$ and $\sum_{R \in O(R')} s(R') = s'(R')$. This is a well-defined strategy profile in G since $R' = R \setminus P_G \subseteq R$ for every $R \in O(R')$ and $R' \in 2_{\neq \emptyset}^{[n] \setminus P_G}$, and by definition of the player mass in G' and G'' . By definition of s , we have that $h_j^s = h_j^{s''}$ for every $j \in P_G$ and $h_j^s = h_j^{s'}$ for every $j \in [n] \setminus P_G$. Therefore, by definition of s'' we have that $h_j^s = h_j^{s''} = h_G$ for every $j \in P_G$, and by definition of s' and by Lemma 5(2), we have that $h_j^s = h_j^{s'} \leq h_{G'} < h_G$ for every $j \in [n] \setminus P_G$. Therefore, we have that $h_j^s \leq h_G$ for every $j \in [n]$.

We complete the proof by showing that s is a Nash equilibrium in G . For every $R \in 2_{\neq \emptyset}^{P_G}$, $k \in \text{supp}(s(R)) \subseteq R$ and $j \in R$, we have by definition of s, s'' that $h_k^s = h_k^{s''} = h_G = h_j^{s''} = h_j^s$. Let $R \in 2_{\neq \emptyset}^{[n] \setminus P_G}$, $k \in \text{supp}(s(R))$ and $j \in R$. By definition of s , we have that $k \in \text{supp}(s'(R \setminus P_G)) \subseteq 2_{\neq \emptyset}^{[n] \setminus P_G}$. If $j \in [n] \setminus P_G$, then $j \in R \setminus P_G$ and by definition of s, s' we

have that $h_k^s = h_k^{s'} \leq h_j^{s'} = h_j^s$; otherwise, i.e. if $j \in P_G$, then by Lemma 5(2) and by definition of s, s', s'' we have that $h_k^s = h_k^{s'} \leq h_G' < h_G = h_j^{s''} = h_j^s$. Either way, $h_k^s \leq h_j^s$ and the proof is complete. \square

Proof of Theorem 2. We prove the theorem by full induction on n .

Let $n \in \mathbb{N}$ and assume that the theorem holds for all smaller natural values of n . Let $G = ((f_j)_{j=1}^n; (\mu^R)_{R \in 2_{\neq \emptyset}^{[n]}})$ be an n -resource selection game and let s, s' be Nash equilibria in G . By Lemma 4(1,2), $h_j^s = h_G = h_j^{s'}$, for every $j \in P_G$. If $P_G = [n]$, then the proof is complete. Otherwise, let $s'', s''' : 2_{\neq \emptyset}^{[n] \setminus P_G} \rightarrow \mathbb{R}_{\geq}^{[n] \setminus P_G}$ be the functions defined by $s_j''(R') \triangleq \sum_{R \in O(R')} s_j(R)$ and $s_j'''(R') \triangleq \sum_{R \in O(R')} s_j'(R)$ for every $j \in [n] \setminus P_G$ (where $O(R')$ is defined as in Lemma 4(4)). By Lemma 4(1,4), s'', s''' are both Nash equilibria in $((f_j)_{j \in [n] \setminus P_G}; (\sum_{R \in O(R')} \mu^R)_{R' \in 2_{\neq \emptyset}^{[n] \setminus P_G}})$, and so, by the induction hypothesis (since $P_G \neq \emptyset$ by Lemma 3), we obtain that $h_j^{s''} = h_j^{s'''}$ for every $j \in [n] \setminus P_G$. Therefore, by Lemma 4(1,4), we have $h_j^s = h_j^{s''} = h_j^{s'''} = h_j^{s'}$ for every $j \in [n] \setminus P_G$ as well, and so $h_j^s = h_j^{s'}$ for every $j \in [n]$, as required. \square

Proof of Corollary 1. We start by proving Part 1. Let s, s' be Nash equilibria in G , and let $R \in 2_{\neq \emptyset}^{[n]}$. By definition of Nash equilibrium and by Theorem 2, we have for every $k \in \text{supp}(s(R))$ and $k' \in \text{supp}(s'(R))$ that $h_k^s = \min_{j \in R} h_j^s = \min_{j \in R} h_j^{s'} = h_{k'}^{s'}$, as required.

We move on to prove Part 2; Let $j \in [n]$. Let $S = \{k \in [n] \mid h_k^s = h_j^s\}$; by Theorem 2, $S = \{k \in [n] \mid h_k^{s'} = h_j^{s'}\}$ as well. Let $\mathcal{R} \triangleq \{R \in 2_{\neq \emptyset}^{[n]} \mid \text{supp}(s(R)) \subseteq S\}$; by Theorem 2 and Part 1, $\mathcal{R} = \{R \in 2_{\neq \emptyset}^{[n]} \mid \text{supp}(s'(R)) \subseteq S\}$ as well. Assume w.l.o.g. that h_j^s is not a plateau height of any of $S \setminus \{j\}$; we therefore have to show that $\mu_k^s = \mu_k^{s'}$ for every $k \in S$. For every $k \in S \setminus \{j\}$, as h_j^s is not a plateau height of f_k , there exists a unique value $\mu_k \in \mathbb{R}_{\geq}$ s.t. $f_k(\mu_k) = h_j^s$. Therefore, and as by definition of S and by Theorem 2 we have that $f_k(\mu_k^s) = h_k^s = h_j^s = h_j^{s'} = h_k^{s'} = f_k(\mu_k^{s'})$ for every $k \in S \setminus \{j\}$, we have that $\mu_k^s = \mu_k = \mu_k^{s'}$ for every $k \in S \setminus \{j\}$. By Part 1, we have that $\sum_{k \in S} \mu_k^s = \sum_{R \in \mathcal{R}} \mu^R = \sum_{k \in S} \mu_k^{s'}$, and so $\mu_j^s = \sum_{R \in \mathcal{R}} \mu^R - \sum_{k \in S \setminus \{j\}} \mu_k^s = \sum_{R \in \mathcal{R}} \mu^R - \sum_{k \in S \setminus \{j\}} \mu_k^{s'} = \mu_j^{s'}$. Therefore, $\mu_k^s = \mu_k^{s'}$ for every $k \in S$ and the proof is complete. \square

Proof of Theorem 3. We begin by proving Part 1 by full induction on n . Let $n \in \mathbb{N}$ and assume that the claim holds for all smaller natural values of n . Let $G = ((f_j)_{j=1}^n; (\mu^R)_{R \in 2_{\neq \emptyset}^{[n]}})$ be an n -resource selection game and let s be Nash equilibrium in G . For every $R \in 2_{\neq \emptyset}^{[n]}$ with $\mu^R \neq 0$, let $h^R \triangleq h_j^s$ for every $j \in \text{supp}(s(R))$. Let $T \subseteq 2_{\neq \emptyset}^{[n]}$ be a coalition and s' be a consumption profile s.t. $s'|_{2_{\neq \emptyset}^{[n]} \setminus T} = s|_{2_{\neq \emptyset}^{[n]} \setminus T}$ and s.t. $h_k^{s'} < h^R$ for every $R \in T$ and $k \in \text{supp}(s'(R))$ s.t. $s'_k(R) > s_k(R)$. We must show that $s' = s$.

We begin by showing that $s'(R) = s(R)$ for every $R \in 2_{\neq \emptyset}^{P^s}$, for P^s as defined in Lemma 4. Assume for contradiction that $s'(R) \neq s(R)$ for some $R \in 2_{\neq \emptyset}^{P^s}$; therefore, $R \in T$. Let $S = \{j \in P^s \mid h_j^{s'} < h_G\} \subseteq P^s$. As $s'(R) \neq s(R)$, there exists $k \in R$ s.t. $s'_k(R) > s_k(R)$ and so $k \in \text{supp}(s'(R))$. Therefore, by definition of s' and by Lemma 4(2), we have that $h_k^{s'} < h_k^s = h_G$ and so $k \in S$; in particular, $S \neq \emptyset$.

For every $j \in S$, by definition of S and by Lemma 4(2), we have that $f_j(\mu_j^{s'}) = h_j^{s'} < h_G = h_j^s = f_j(\mu_j^s)$; therefore, as f_j is nondecreasing we have that $\mu_j^{s'} < \mu_j^s$ for every such j . Therefore, as $S \neq \emptyset$, $\sum_{j \in S} \mu_j^{s'} < \sum_{j \in S} \mu_j^s$. By definition of consumption profile and by Lemma 4(3), $\sum_{j \in P^s} \mu_j^{s'} \geq \sum_{R' \in 2_{\neq \emptyset}^{P^s}} \mu^{R'} = \sum_{j \in P^s} \mu_j^s$. By both of these, $\sum_{j \in P^s \setminus S} \mu_j^{s'} > \sum_{j \in P^s \setminus S} \mu_j^s$, and so

there exists $j \in P^s \setminus S$ s.t. $\mu_j^{s'} > \mu_j^s$; hence, there exists $R' \in 2_{\neq \emptyset}^{[n]}$ s.t. $s'_j(R') > s_j(R')$. Therefore, $R' \in T$, however, by definition of S and as f_j is nondecreasing, we have that

$$h_j^{s'} = f_j(\mu_j^{s'}) \geq f_j(\mu_j^s) = h_j^s = h_G \geq h^{R'} \quad (2)$$

(as $s'_j(R') > 0$, $h^{R'}$ is well defined), even though $s'_j(R') > s_j(R')$ — a contradiction. Therefore, $s'(R) = s(R)$ for every $R \in 2_{\neq \emptyset}^{P^s}$ and so we may assume w.l.o.g. that $T \cap 2_{\neq \emptyset}^{P^s} = \emptyset$. By definition of s' , by definition of P^s and by Lemma 4(3), we thus obtain that $s'_j \equiv s_j$ for every $j \in P^s$.

If $P^s = [n]$, then the proof is complete. Otherwise, define $s'' : 2_{\neq \emptyset}^{[n] \setminus P^s} \rightarrow \mathbb{R}_{\geq}^{[n] \setminus P^s}$ by $s''_j(R') \triangleq \sum_{R \in O(R')} s_j(R)$ for every $j \in [n] \setminus P^s$, where $O(R')$ is defined for every $R' \in 2_{\neq \emptyset}^{[n] \setminus P^s}$ as in Lemma 4(4). By Lemma 4(4), s'' is a Nash equilibrium in the game $G' \triangleq ((f_j)_{j \in [n] \setminus P^s}; (\sum_{R \in O(R')} \mu^R)_{R' \in 2_{\neq \emptyset}^{[n] \setminus P^s}})$, and $h_j^{s''} = h_j^s$ for every $j \in [n] \setminus P^s$. For every $R' \in 2_{\neq \emptyset}^{[n] \setminus P^s}$ with $\mu^{R'} \neq 0$, let $h^{R'} \triangleq h_j^{s''}$ for every $j \in \text{supp}(s''(R'))$; by definition, $h^{R'} = h^R$ for every $R' \in 2_{\neq \emptyset}^{[n] \setminus P^s}$ and $R \in O(R')$ s.t. $\mu^R \neq 0$.

Similarly, define $s''' : 2_{\neq \emptyset}^{[n] \setminus P^s} \rightarrow \mathbb{R}_{\geq}^{[n] \setminus P^s}$, by $s'''_j(R') \triangleq \sum_{R \in O(R')} s'_j(R)$ for every $j \in [n] \setminus P^s$. As $s'_j \equiv s_j$ for every $j \in P^s$, we have that, similarly to the proof of Lemma 4(4), s''' is a strategy profile in G' and $h_j^{s'''} = h_j^{s''}$ for every $j \in [n] \setminus P^s$. Define $T' \triangleq \{R \setminus P^s \mid R \in T\} \in 2_{\neq \emptyset}^{[n] \setminus P^s}$. By definition of T' , we have that $s'''|_{2_{\neq \emptyset}^{[n] \setminus P^s} \setminus T'} = s''|_{2_{\neq \emptyset}^{[n] \setminus P^s} \setminus T'}$ and that $h_k^{s'''} = h_k^{s''} < h^R = h^{R'}$ for every $R' \in T'$ and $k \in \text{supp}(s'''(R'))$ s.t. $s'''_k(R') > s''_k(R')$, where $R \in O(R')$ s.t. $k \in \text{supp}(s'(R))$ and $s'_k(R) > s_k(R)$ (there exists such R by definition of R'). By the induction hypothesis (since $P^s \neq \emptyset$ by definition), $T' = \emptyset$, and so $T = \emptyset$ and the proof of Part 1 is complete.

The proof of Part 2 is very similar; the main difference is that in Eq. (2) we would have, by h_j^s not being a plateau height of f_j , that $h_j^{s'} = f_j(\mu_j^{s'}) > f_j(\mu_j^s) = h_j^s = h_G \geq h^{R'}$. The remaining trivial differences between Parts 1 and 2 are left to the reader. \square